Additive Representations on a Simplex

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Abstract

We characterize additive representations on product spaces with an empty interior such as simplexes and certain homeomorphisms thereof. Previously, all additive representation theorems only applied to spaces in which any coordinate can be changed without changing any of the other coordinates. We identify a novel preference condition that is necessary and sufficient for the existence of additive representations. Our results provide a characterization of utilitarianism on the Pareto frontier of a cake division problem.

Keywords: Additive Separability, Simplex, Preferences, Utility Representation, Utilitarianism

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1 Introduction

Additive representations of preferences are ubiquitous in economic theory. For example, preferences over consumption goods are often assumed to be additively separable in each of the goods. Expected utility preferences over lotteries are additively separable (and even linear) in the probabilities of the prizes. Utilitarian social planner preferences over allocations are additively separable across the individuals affected by policies.

Debreu (1959) first characterized additive representations for product spaces. Chateauneuf and Wakker (1993) extended these results to open subsets of product spaces.1 Presently existing characterizations all require the space of alternatives to have a nonempty interior. Thus, these axiomatic characterizations require that each dimension can be changed without changing any of the other dimensions. However, this is not always plausible. In the case of a policy maker’s preferences over income allocations to individuals, this means that preferences need to also be formed over allocations in which one individual’s income is reduced without increasing any other individual’s income. In practice, preferences over such wasteful allocations may not be observable or only be observable in hypothetical choices. Thus, if a policy maker’s preferences are only elicited on the Pareto frontier, the existing additive representation theorems would not apply. This problem is even more severe in spaces of lotteries – which naturally form a subset of a product space – since lotteries with probabilities that do not sum to one are not meaningful. We cannot reduce the probability of some outcome without increasing the probability of another. In this case even hypothetical choices would not allow us to recover a product space structure to which existing additive representation theorems apply.

In the present paper we address this problem by providing an axioma-
tization of additively separable representations on simplexes and a class of hypersurfaces. In economics, simplexes are commonly used as lottery spaces with finitely many states, as sets of consumption bundles given a fixed expenditure, as relative abundances in diversity measurements, or as allocations in cake division problems. Extending additive representation theorems to simplexes therefore allows for a broad range of additional applications of additive representation analysis. Possible applications of our representation theorem include utilitarian preferences when preferences are restricted to the Pareto frontier, measures of diversity and inequality, and von Neumann–Morgenstern

1For a more extensive review of additive representations, see Wakker (1988), Wakker (1989).
preferences on lotteries with arbitrary probability weighting.

Additive representations are much harder to obtain on spaces with nonempty interior than on the usual product spaces. The key observation is that the standard preference separability axioms such as \((x_i, x_{-i}) \succsim (x'_i, x_{-i}) \iff (x_i, x'_{-i}) \succsim (x'_i, x_{-i})\) are greatly weakened if preferences are restricted to a domain with an empty interior. In this domain, standard separability of a single coordinate from the remaining coordinates does not restrict preferences at all since no single coordinate can be changed without changing at least one other coordinate. Preference separability of several coordinates from the remaining coordinates still restricts preferences somewhat but is severely restricted due to this requirement.

In addition to the usual separability conditions, we therefore employ a novel condition, comeasurability, which is necessary and sufficient for the existence of an additive representation of preferences. Comeasurability is a cardinal comparability condition that guarantees that the separability conditions that hold for different subsets of the preference domain combine into an overall additive representation.

The paper continues as follows. Section 2 discusses the problem of additive representations on simplexes informally. We then introduce the basic notation in Section 3. We characterize additive representations on simplexes and a class of homeomorphisms thereof in Section 4 before concluding in Section 5.

2 Weakness of Separability Conditions on Simplexes

There exist a variety\(^2\) of additive representation theorems, i.e., characterizations of conditions on preferences such that for alternatives \(x = (x_1, \ldots, x_n)\) and \(y = (y_1, \ldots, y_n)\) we have that \(x \succeq y\) if and only if \(\sum_{i=1}^{n} u_i(x_i) \geq \sum_{i=1}^{n} u_i(y_i)\). These representation theorems however commonly require that the preference is defined (at least) on a subset of a product space with a nonempty interior. This is for example fulfilled by a volume within a three-dimensional space but not by a surface. The key difference is that in the volume for every interior point we can find a small change in any dimension without changing the other dimensions. For example, the coordinate independence axiom (Wakker, 1989) requires that if two alternatives agree on one coordinate, then the preference

between the two alternatives remains unchanged if we change only this single coordinate. Coordinate independence of all dimensions— together with some regularity and continuity conditions— is sufficient to guarantee the existence of an additive representation. The coordinate independence axiom however does not restrict preferences if there is no second alternative that only differs by a single coordinate. Yet, in many decision domains in economics we cannot change one coordinate of a space without changing one of the other coordinates. For example in a lottery, it is not meaningful to increase the probability of some prize without decreasing the probability of another prize. In such spaces, even if a decision maker would fulfill coordinate independence or other separability conditions of the standard additive representation theorems, we could not conclude that the decision maker’s preference relation has an additive representation.

On the converse, clearly the standard preference separability conditions are necessary for the existence of an additive representation and thus any changes in a subset of dimensions yield either an improvement or worsening independent of the remaining dimensions. What is special on a simplex is that these separability conditions only hold for a fixed total amount assigned to each subset of coordinates. For example in additive representations on a four dimensional simplex, the preference over the first two coordinates \( x_1 \) and \( x_2 \) are separable from the preference over the latter two coordinates, \( x_3 \) and \( x_4 \), but separate changes in the first two coordinates are only possible as long as we fix the sum of these coordinates, \( x_1 + x_2 = 1 - x_3 - x_4 \). This is because increasing \( x_1 \) without changing \( x_3 \) and \( x_4 \) requires decreasing \( x_2 \) by an equal amount. Preference separability thus becomes:

\[
(x_1, x_2, x_3, x_4) \succ (x'_1, x'_2, x_3, x_4) \iff (x_1, x_2, x'_3, x'_4) \succ (x'_1, x'_2, x'_3, x'_4)
\]

and thus on a four dimensional simplex—which is a tetrahedron—preference separability effectively only restricts preferences on every rectangular slice fulfilling \( x_1 + x_2 = c \) for some constant \( c \). Standard additive representation theorems would then only yield representations such as \( h(f(x_1, x_1 + x_2) + g(x_3, x_1 + x_2), x_1 + x_2) \). Even if we obtain several such representations for different combinations of separable dimensions, we would need a method to combine these into an overall additive representation.

We solve this problem in the following way. First, we employ the results

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3Further conditions than just separability are of course needed.
of Qin and Rommeswinkel (2018) to obtain quasi-separable representations holding fixed the total amount allocated to two dimensions, i.e., representations of the form \( f(x_1, x_1 + x_2) + g(x_3, x_1 + x_2). \) The results of Qin and Rommeswinkel (2018) therefore allow us to avoid the additional complication of the \( h(\ldots, x_1 + x_2) \) function. Second, we combine these representations into an additive representation by solving a novel functional equation, relying on an existing functional equation from Hosszú (1971). Both the first and second step makes use of a new axiom, comeasurability, which is a cardinal comparability axiom.

3 Model and notation

For a finite set of dimensions \( \mathcal{O} \), let \( \overline{P} = \{ s : \mathcal{O} \rightarrow [0,1] | \sum_{o \in \mathcal{O}} s(o) = 1 \} \) be an \( n-1 \)-dimensional (regular) simplex where \( n = |\mathcal{O}|. \) Let \( P = \{ s : \mathcal{O} \rightarrow (0,1) | \sum_{o} s(o) = 1 \} \) be its relative interior. Simplexes are spaces with a natural algebraic structure, convex combinations. This allows to express certain axioms on preferences more conveniently. For any two \( s, s' \in \overline{P}, \) we define their convex combination \( s'' = as \oplus (1-a)s' \) as the element of \( \overline{P} \) that fulfills \( s''(o) = as(o) + (1-a)s'(o). \) The support of an element \( s \in \overline{P} \) is the set \( \text{supp}(s) = \{ o \in \mathcal{O} | s(o) > 0 \}. \)

We can also treat the simplex as a vector space and denote \( P = \{ x \in (0,1)^n | \sum_{i=1}^n x_i = 1 \} \) instead for some given enumeration of the elements in \( \mathcal{O}. \) We will use the convention that we reserve the letter \( x \) when a generic element of \( P \) is understood as a vector and the letter \( s \) when it is understood as a function. In some cases it will be convenient to denote that \( x =_\mathcal{O} x' \) if for all \( i, x_i = x'_i, \) i.e., if \( x \) and \( x' \) are identical in the dimensions \( \mathcal{O}.

4 Axiomatization

In this section, we characterize the following representation of preferences.

**Definition 1** (Additive Representation). A relation \( \succsim \) on a set \( S \) has an additive representation if there exists a function \( U : S \rightarrow \mathbb{R} \cup \{-\infty, \infty\} \) and functions
$u_o : [0,1] \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$ such that for all $s, s' \in S$

$$s \succeq s' \iff U(s) \geq U(s') \text{ and } U(s) = \sum_{o \in O} u_o(s(o)).$$

The representation is finite if $U(s) \in \mathbb{R}$ for all $s \in S$.

**Example.** Let $O$ be a set of individuals and $S$ be the set of shares of a unit of a cake that can be allocated to these individuals. If we normalize the amount of the resource to 1, we obtain $\sum_{o \in O} s(o) = 1$ as a constraint. Therefore, the space on which the preferences are defined is an $n - 1$-dimensional simplex with each dimension $o$ representing the share $s(o)$ individual $o$ receives. In such a cake division problem an additive representation can for example be interpreted as utilitarianism. Each $u_o$ is in this interpretation the utility function of individual $o$.

**End of example.**

We allow our representation on $\mathcal{P}$ to take values in the extended reals. In additive representation theorems on product spaces, the combination of essentiality conditions (i.e., nontriviality of the conditional preference with respect to some dimensions) together with preference separability preclude the value of any of the coordinates from becoming infinite. The reason is that if one coordinate is infinite, then the other coordinates do not matter when combined with the infinite utility coordinate. Separability of preferences then implies that these other coordinates never matter. Because our preferences are defined on a subset of a product space, in our representation theorem we can only guarantee the representation to be finite in the interior but not on the boundary. We employ the following essentiality condition:

**Definition 2 (⊕-Strong Essentiality).** $\succsim$ fulfills strong essentiality with respect to a pair of dimensions $O = \{i,j\}$ if for all $\alpha \in (0,1]$ there exist some $s,s',s''$ such that $\text{supp}(s) \subseteq O$, $\text{supp}(s') \subseteq O$, and $\text{supp}(s'') \cap (O \cup Q) = \emptyset$,

$$as \oplus (1-\alpha)s'' \succ as' \oplus (1-\alpha)s''$$

Strong essentiality requires that for every relative amount $\alpha$ allocated to two dimensions, there exist different allocations between the two dimensions that are not indifferent. For expected utility preferences, the independence axiom together with a weak nontriviality condition implies strong essentiality. Since
we employ a weaker independence condition, we must directly require the strong essentiality condition to hold for some dimensions. As an alternative which does not rely on convex combinations and which applies to surfaces in general, we present the following formulation:

**Definition 3 (Strong Essentiality).** \( \succ \) fulfills strong essentiality with respect to a pair of dimensions \( O = \{i, j\} \) if for all \( x \) there exists some \( x' \) with \( x = O - O x' \) such that

\[ x \not\sim x'. \]

**Example.** In the example of the cake division problem, suppose the set of individuals consists of Anna, Bob, Charlie, and Dana. Let’s say a policy maker’s preferences can be represented by weighted utilitarianism with identical, linear utilities, i.e., \( \sum_{o \in O} w(o)s(o) \). If the weights of Charlie and Dana are equal but distinct from Anna and Bob’s, then the preference fulfills strong essentiality with respect to \{Anna, Charlie\} and \{Anna, Dana\} but not with respect to \{Charlie, Dana\}. The main representation theorem of this paper requires the relation to be pairwise strongly essential with respect to at least four dimensions of the support.  

End of example.

Since in a simplex we have convex combinations available, we formulate our preference separability condition as a weakening of the classical von Neumann-Morgenstern independence axiom. First, we fix the support of the convex combinations of elements and require these to be disjoint for independence to hold. Thus, when we mix two elements with identical support, we cannot expect any consistency of its relative ranking with the ranking of its constituent elements. Second, we disallow scaling of the relative weight of the convex combinations. The classical independence axiom generates parallel indifference curves by making preference scale invariant with respect to the relative weight of the convex combinations.

**Definition 4 (⊕-Separability).** A relation fulfills separability if whenever \((supp(s) \cup supp(s')) \cap (supp(s'') \cup supp(s''')) = \emptyset \) and

\[ as \oplus (1 - a)s'' \succeq as' \oplus (1 - a)s''', \]

then,

\[ as \oplus (1 - a)s'''' \succeq as' \oplus (1 - a)s''''. \]
This formulation using convex combinations is equivalent to the standard notion of separability of preferences. An alternative formulation of separability which does not employ convex combinations and which applies to surfaces in general is therefore:

**Definition 5 (Separability)**. A relation \( \succsim \) fulfills separability if for all \( O \subset \emptyset \) and all \( x, x', x'', x''' \) such that \( x =_O x'' \), \( x =^O_{O-O} x' \), \( x'' =_O x''' \), and \( x'' =^O_{O-O} \),

\[
x \succsim x' \iff x'' \succsim x'''
\]

While we could work solely with the latter formulation, the first formulation highlights that in a simplex, separability is much weaker than in a product space because the total amount \( \alpha \) on dimensions \( \text{supp}(s) \cup \text{supp}(s') \) is fixed.

**Example.** In a cake division problem in which Anna, Bob, Charlie, and Dana share a fixed amount of a resource, whenever we want to change let’s say Anna’s share, we also need to change the share of at least one other individual. Thus, standard coordinate independence can only be expressed with respect to changes of at least two coordinates. Our separability axiom guarantees that for a fixed total share \( \alpha \) allocated to Anna and Bob, changes in their relative allocation are either an improvement or a worsening irrespective of the relative allocation of the remaining share between Charlie and Dana. This is equivalent to saying that reallocating some of the absolute share of Anna to Bob is an improvement or worsening irrespective of the absolute shares of Charlie and Dana. In a product space, together with an additional condition such as triple cancellation or the Reidemeister condition, this would be sufficient to guarantee a representation that is additively separable between the absolute shares of Anna and Bob from the shares of Charlie and Dana. However, in a simplex, this would only guarantee an additively separable representation for the fixed share \( \alpha \) that Anna and Bob receive. Separability therefore needs to be supplemented with a stronger condition to guarantee the existence of an additive representation.

End of example.

In addition, we require:

**Definition 6.** Comeasurability:

A relation \( \succsim \) fulfills comeasurability if whenever \( (\text{supp}(s_1) \cup \text{supp}(s'_1)) \cap (\text{supp}(s_2) \cup \text{supp}(s'_2)) \)
\( \text{supp}(s'_2) = \emptyset, (\text{supp}(\bar{s}_1) \cup \text{supp}(s'_1)) \cap (\text{supp}(\bar{s}_2) \cup \text{supp}(s'_2)) = \emptyset \) and

\[
\begin{align*}
  a s_1 \oplus (1 - a)s_2 & \sim a'\bar{s}_1 \oplus (1 - a')\bar{s}_2 \\
  a s'_1 \oplus (1 - a)s_2 & \sim a'\bar{s}'_1 \oplus (1 - a')\bar{s}_2 \\
  a s_1 \oplus (1 - a)s'_2 & \sim a'\bar{s}_1 \oplus (1 - a')\bar{s}'_2,
\end{align*}
\]

then,

\[
  a s'_1 \oplus (1 - a)s'_2 \sim a'\bar{s}'_1 \oplus (1 - a')\bar{s}'_2
\]

Comeasurability is a cardinal comparability condition which states that starting from a convex combination of \( s_1 \) and \( s_2 \) and an indifferent convex combination of \( \bar{s}_1 \) and \( \bar{s}_2 \), the values of equivalent improvements of \( s_1 \) and \( \bar{s}_1 \) and the values of equivalent improvements of \( s_2 \) and \( \bar{s}_2 \) are independent of another. Thus, applying all four improvements yields two convex combinations that are still indifferent.

On surfaces more generally, comeasurability can be alternatively formulated as the following condition.

**Definition 7 (Comeasurability).** A relation \( \succcurlyeq \) fulfills comeasurability if for all \( O, Q \subset \emptyset \) and all \( x, x', x'', x''' \) and \( \bar{x}, \bar{x}', \bar{x}'', \bar{x}''' \) such that \( x =_O x' \), \( \bar{x} =_Q \bar{x}' \), \( x =_{O-O} x'' \), \( \bar{x} =_{Q-Q} \bar{x}'' \), \( x' =_O x''' \), \( \bar{x}' =_Q \bar{x}''' \), \( x'' =_{O-O} x''' \), and \( \bar{x}'' =_{Q-Q} \bar{x}''' \),

\[
\begin{align*}
  x & \sim \bar{x} \\
  x' & \sim \bar{x}' \\
  x'' & \sim \bar{x}'' \\
  \Rightarrow \quad x''' & \sim \bar{x}'''.
\end{align*}
\]

**Example.** In the cake division problem, we may consider relative shares \( s_1 \) to Anna and Bob and relative shares \( s_2 \) Charlie and Dana. Let’s say we overall allocate a share \( \alpha \) to Anna and Bob. We may now consider improvements of either the allocation between Anna and Bob, \( s'_1 \), or to Charlie and Dana, \( s'_2 \). Separability guarantees that if \( s'_2 \) is an improvement over \( s_2 \), then it is an improvement irrespective whether Anna and Bob share their part of the cake by \( s_1 \) or by \( s'_1 \). However, it does not fix the scale of the improvement, i.e., combining the improvement of \( s_1 \) to \( s'_1 \) with the improvement of \( s_1 \) to \( s'_2 \) may be superadditive and yield and extraordinarily good allocation. Comeasurability guarantees that this is not the case by requiring that all improvements are cardinally comparable.

End of example.
Comeasurability plays a threefold role in our proof. First, it functions as the so-called Reidemeister condition to construct an additive representation whenever we hold fixed the total amounts allocated to subsets of the support $\text{supp}(s_1)$ and $\text{supp}(s_2)$, let’s say. Comeasurability reduces to the Reidemeister condition when $\text{supp}(s_1) = \text{supp}(\bar{s}_1)$ and $\text{supp}(s_2) = \text{supp}(\bar{s}_2)$ and $\alpha = \alpha'$. This allows us to generate an additive representation $f_{\text{supp}(s_1)}(s_1, \alpha) + g_{\text{supp}(s_2)}(s_2, 1 - \alpha)$ on the subset of all elements with total sum $\alpha$ on $\text{supp}(s_1)$. Second, it ensures that all these additive representations are cardinally comparable with each other for different values of $\alpha$ and that therefore the preference can be represented on the entire space by $f_{\text{supp}(s_1)}(s_1, \alpha) + g_{\text{supp}(s_2)}(s_2, \alpha)$. Third, it guarantees cardinal comparability of additive representations with different dimensions. If $\alpha s_1 \oplus (1 - \alpha)s_2 = \beta \bar{s}_1 \oplus (1 - \beta)\bar{s}_2$, then comeasurability ensures that $f_{\text{supp}(s_1)}(s_1, \alpha) + g_{\text{supp}(s_2)}(s_2, 1 - \alpha) = f_{\text{supp}(\bar{s}_1)}(\bar{s}_1, \beta) + g_{\text{supp}(\bar{s}_2)}(\bar{s}_2, 1 - \beta)$.

**Definition 8 (Continuity).** $\succcurlyeq$ is a continuous relation on a set $S \subseteq \mathbb{R}^n$ if for all $s \in S$ the sets $\bar{S}(s) = \{s' \in S : s' \succ s\}$ and $\bar{S}(s) = \{s' \in S : s \succ s'\}$ are open in the subspace topology on $S$.

In the appendix we prove the following theorem.

**Theorem 1.** Suppose $\succcurlyeq$ is a preference relation on $\bar{P}$. Let at least 4 dimensions be pairwise $\oplus$-strongly essential. Then, the following statements are equivalent.

- $\succcurlyeq$ fulfills continuity, $\oplus$-separability, and $\oplus$-comeasurability.

- $\succcurlyeq$ has an additive representation $U$ that is finite on $P$.

The function $U$ and its additive components are unique up to joint linear and separate additive transformations.

This result is similar to additive representation theorems on product spaces. The central difference to the axioms used in additive representation theorems is the additional use of comeasurability.

**Example.** Theorem 1 provides an axiomatization for utilitarianism in a cake division problem. As can be seen, preference separability conditions across individuals are not sufficient to guarantee that the preference of the policy maker can be represented by a utilitarian representation. Instead, we need to explicitly impose via comeasurability that all utility improvements are cardinally comparable.

End of example.
Theorem 1 can be generalized to a class of surfaces without the linear structure that a simplex has. This is because the following equivalence holds:

Remark 1. Suppose for a space $S'$ with preference $\succeq'$ we can find a homeomorphism $h : S' \rightarrow P$ such that $h(s') = h(x'_1, \ldots, x'_n) = (h_1(x'_1), \ldots, h_n(x'_n))$, each $h_i$ is a homeomorphism, and

$$s' \succeq' \bar{s}' \iff h(s') \succeq h(\bar{s}')$$

Then,

1. $\succeq$ fulfills $\oplus$-strong essentiality with respect to a pair of dimensions if and only if $\succeq'$ fulfills strong essentiality with respect to this pair of dimensions,

2. $\succeq$ fulfills $\oplus$-separability if and only if $\succeq'$ fulfills separability,

3. $\succeq$ fulfills $\oplus$-comeasurability if and only if $\succeq'$ fulfills comeasurability,

4. $\succeq$ fulfills continuity if and only if $\succeq'$ fulfills continuity.

This remark is straightforward once we observe that the $\oplus$-versions of the axioms only rely on the use of convex combinations of elements with a disjoint support. The equivalence of continuity follows directly from $h$ being a homeomorphism. Notice that we require a separate homeomorphism for every dimension which is less general than only requiring a homeomorphism from the simplex to the surface.

Example. In our cake division problem, we could assume that each individual $i$ has a resource cost $h_i(x_i)$ of producing $x_i$ of a consumption good. If each $h_i$ is continuous and strictly increasing, Remark 1 applies and Theorem 1 holds for $S' = \{ x \in \prod_{i=1}^n X_i : h_n(x_n) = 1 - \sum_{i=1}^{n-1} h_i(x_i) \}$. This allows to extend our characterization of utilitarianism to the case in which individuals have exogenously given different abilities to make use of the resource. The case of linear $h_i$ functions shows that our result for a regular simplex indeed also holds for a non-regular simplex.

End of example.

5 Concluding Remarks

Both additively separable representations as well as simplexes are used in many areas of economics. In this paper we have closed the gap in the literature that
representation theorems of additively separable representations did not apply to simplexes. In addition to separability we have identified comeasurability as a necessary and sufficient condition for the existence of additively separable representations on simplexes. Separability plays the role of preference separability across dimensions of the simplex. Comeasurability is a cardinal comparability condition for changes across different dimensions of the simplex.

**APPENDIX A  SOLUTION TO** $f(x, y) - g(x, z) = h(y, x + z) - i(z, x + y)$

We first provide the solution to a functional equation that will be useful in the remainder of the paper.

**Lemma 1.** Let $S, +$ be a cancellative abelian monoid and let $\bar{f}, \bar{g}, f$ and $g$ be real valued functions defined on $S^2$ and satisfy the equation

$$\bar{f}(x_3, x_1 + x_2) + \bar{g}(x_1, x_2) = f(x_2, x_1 + x_3) + g(x_1, x_3)$$

for all $x_1, x_2, x_3$ in $S$. Then, $f(x_2, x_1 + x_3) + g(x_1, x_3) = v_{123}(x_1 + x_2 + x_3) + v_1(x_1) + v_2(x_2) + v_3(x_3)$. In particular, $f(a, b) = a_1(a) + a_2(b) + a_3(a + b)$.

**Proof.** The functional equation to be solved is

$$\bar{g}(x_1, x_2) = f(x_2, x_1 + x_3) + g(x_1, x_3) - \bar{f}(x_3, x_1 + x_2).$$

We set $x_3 = 0$ and define $\bar{u}_1(x_1) = g(x_1, 0)$ and $\bar{u}(x_1) = \bar{f}(0, x_1)$ to obtain:

$$\bar{g}(x_1, x_2) = f(x_2, x_1) + \bar{u}_1(x_1) + \bar{u}_3(x_1 + x_2).$$

By a symmetric argument with $x_2 = 0$, we obtain

$$g(x_1, x_3) = \bar{f}(x_3, x_1) + u_1(x_1) + u_3(x_1 + x_3).$$

---

\[4\] In the remainder of the proof, we will omit stating that equations such as (A) hold for all $x_1, x_2, x_3$. It will be clear from the context whether a variable is a free variable or not.
Inserting Equation (A) and Equation (A) into Equation (A), we obtain

\[ f(x_2, x_1 + x_3) + \bar{f}(x_3, x_1) + u_1(x_1) + u_3(x_1 + x_3) \]
\[ = \bar{f}(x_3, x_1 + x_2) + f(x_2, x_1) + \bar{u}_1(x_1) + \bar{u}_3(x_1 + x_2). \]

Let \( x_1 = 0 \) in Equation (A). Then, we get the following equation between \( \bar{f} \) and \( f \),

\[ \bar{f}(x_3, x_2) = f(x_2, x_3) + A_1(x_2) + A_2(x_3) \]

for some suitably defined functions \( A_1, A_2 \). Inserting this result into Equation (A), we get

\[ f(x_2, x_1 + x_3) + f(x_1, x_3) + U_1(x_1) + U_2(x_3) + U_3(x_1 + x_3) = \]
\[ f(x_1 + x_2, x_3) + f(x_2, x_1) + \bar{U}_1(x_1) + \bar{U}_2(x_2) + \bar{U}_3(x_1 + x_2). \]

in which we have replaced the sums of functions \( u_1, \ldots, u_3, A_1, A_2 \) by \( U_1, \ldots, \bar{U}_3 \).

We want to characterize the function \( f \), for any \((x_1, x_2) \in S^2\). Gathering terms and relabeling, we have

\[ f(x_1, x_2) = f(x_1, x_2 + x_3) + f(x_2, x_3) - f(x_1 + x_2, x_3) \]
\[ + v_1(x_2) + v_2(x_1) + v_3(x_3) + v_{12}(x_1 + x_2) + v_{13}(x_2 + x_3). \]

Our next goal is to prove \( f(x, x_2) = a_1(x) + a_2(x_2) + a_3(x + x_2) \). To achieve this, we provide the following lemma:

**Lemma 2.** Let \( g : S^2 \to \mathbb{R} \). Then, \( g(x_1, x_2) = g_1(x_1) + g_2(x_2) \) if and only if \( g(x_1', x_2') - g(x_1', 0) - g(0, x_2') + g(0, 0) = 0 \) for all \( x_1', x_2' \).

**Proof.** \( \Rightarrow \): If \( g(x_1, x_2) = g_1(x_1) + g_2(x_2) \), then \( g(x_1', x_2') - g(x_1', 0) - g(0, x_2') + g(0, 0) = g_1(x_1') + g_2(x_2') - g_1(x_1') - g_2(0) - g_1(0) - g_2(x_2') + g_1(0) + g_2(0) = 0. \)

\( \Leftarrow \): Suppose \( g(x_1, x_2) \) satisfies the condition \( g(x_1', x_2') - g(x_1', 0) - g(0, x_2') + g(0, 0) = 0 \). Then, we define the functions \( g_1(x_1) \equiv g(x_1, 0) \) and \( g_2(x_2) \equiv g(0, x_2) - g(0, 0) \). By this condition, \( g(x_1, x_2) = g(0, x_2) + g(x_1, 0) - g(0, 0) = g_1(x_1) + g_2(x_2) \). By Lemma 2, \( f(x_1, x_2) = a_1(x_1) + a_2(x_2) + a_3(x_1 + x_2) \) if and only if \( f(x_1, x_2) - f(x_1, 0) - f(0, x_2) - f(0, 0) = a_3(x_1 + x_2) - a_3(x_1) - a_3(x_2) + a_3(0) \). Therefore, we define

\[ G(x_1, x_2) \equiv f(x_1, x_2) - f(x_1, 0) - f(0, x_2) - f(0, 0). \]
We choose $c = 0$ in $N(x_1 + x_2, x_3), N(x_1, x_2 + x_3)$ and $N(x_2, x_3)$, and $c = x_3$ in $N(x_1, x_2)$ to obtain $N(x_1 + x_2, x_3) + N(x_1, x_2) = N(x_1, x_2 + x_3) + N(x_2, x_3)$. By Hosszú, 1971, $N(x_1, x_2) = B(x_1, x_2) + a(x_1 + x_2) - a(x_1) - a(x_2)$, in which $B(x_1, x_2)$ is a skew-symmetric biadditive function. Since $N(0,0) = N(x_1,0) = N(0,x_2) = 0$, $B(x, x_2) = B(0,0) = a(0) = 0$. Thus, the function $f$ has the functional form

$$f(a, b) = a_1(a) + a_2(b) + a_3(a + b).$$

To show that $f(x_2, x_1 + x_3) + g(x_1, x_3)$ has the desired functional form, we substitute $a = x_1 + x_2$ in Equation (A). Then we obtain

$$U_1(x_1) + U_2(x_2) + U_3(x_3) + U_4(x_1 + x_2)
= \bar{U}_1(x_1) + \bar{U}_2(x_2) + \bar{U}_3(x_3) + \bar{U}_4(x_1 + x_3).$$

Letting $x_3 = 0$, we obtain that $U_4(x_1 + x_2)$ is additively separable in variables $x_1$ and $x_2$. Similarly, letting $x_2 = 0$, $\bar{U}_4$ is additively separable in $x_1$ and $x_3$. The
Appendix B  Proof of Theorem 1

Necessity of the axioms is straightforward, we prove sufficiency. Our proof proceeds in the following steps.

1. We provide a homeomorphism from $P$ to an open subset of a product space $\hat{S}$.

2. We show that the assumptions of Theorem 1 of Qin and Rommeswinkel (2018) are fulfilled and conclude that a quasi-separable representation exists on $\hat{S}$. The representation is additively separable between dimensions $i, j$ and $k, l$ when holding fixed the total amount allocated to dimensions $i, j$.

3. We repeat the previous step, this time to obtain a quasi-separable representation between dimensions $i, k$ and $j, l$.

4. Comeasurability guarantees that the representations are cardinally comparable. This yields the functional equation solved in Lemma 1.

Proof. Define $\hat{S} = \{(x, y_1, \ldots, y_{n-3}, z) \in (0, 1)^{n-1} : 1 - x > z > \sum_{j=1}^{n-3} y_j\} \subset \hat{X} \times \hat{Y} \times \hat{Z}$ with $\hat{X} = (0, 1), \hat{Y} = (0, 1)^{n-3}$, and $\hat{Z} = (0, 1)$. Let the $n$th dimension be one of the four dimensions that are pairwise strongly essential. We define a homeomorphism

$$\phi_i : (S, t^R_S) \rightarrow (\hat{S}, t^R_{\hat{S}})$$

such that:

$$\phi_i(x_1, \ldots, x_n) = (x_i, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n-2}, 1 - x_i - x_n).$$

In other words, the homeomorphism reorders the coordinates such that the $i$th is the first dimension, removes the $n-1$th and $n$th coordinates and adds a coordinate that is equal to the sum of all coordinates minus the $i$th and the $n$th. Thus, under the homeomorphism $\phi_i$, $x = x_i$, $y = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$, and $z = 1 - x_i - x_n$.

Next, for all with $n$ strongly essential dimensions $i$, define $\preceq_i$ as $s \succeq_i s'$ if and only if $\phi_i^{-1}(s) \succeq \phi_i^{-1}(s')$. Note, $\preceq_i$ is continuous if $\succeq$ is continuous since $\phi_i$ is a homeomorphism. Separability of $\preceq_i$ implies conditional independence of $X$...
and $Y$ given $Z$ according to the definitions of Qin and Rommeswinkel (2018). For example in four dimensions and $i = 2$, if $x_2 + x_4 = x'_2 + x'_4$, then:

$$(x, y, z) \succ (x', y, z)$$

$$\Leftrightarrow (x_2, x_1, 1 - x_2 - x_4) \succ (x'_2, x_1, 1 - x'_2 - x'_4)$$

$$\Leftrightarrow (x_1, x_2, x_3, x_4) \succ (x'_1, x'_2, x'_3, x'_4)$$

$$\Leftrightarrow (1 - z)(0, \frac{x_2}{1 - z}, 0, \frac{x_4}{1 - z}) \oplus z(\frac{x_1}{z}, 0, \frac{x_3}{z}, 0) \succ (1 - z)(0, \frac{x'_2}{1 - z}, 0, \frac{x'_4}{1 - z}) \oplus z(\frac{x'_1}{z}, 0, \frac{x'_3}{z}, 0)$$

$$\Leftrightarrow (1 - z)(0, \frac{x_2}{1 - z}, 0, \frac{x_4}{1 - z}) \oplus z(\frac{x'_1}{z}, 0, \frac{x'_3}{z}, 0) \succ (1 - z)(0, \frac{x'_2}{1 - z}, 0, \frac{x'_4}{1 - z}) \oplus z(\frac{x'_1}{z}, 0, \frac{x'_3}{z}, 0)$$

$$\Leftrightarrow (x'_1, x'_2, x'_3, x'_4) \succ (x_1, x_2, x_3, x_4)$$

$$\Leftrightarrow (x, y, z) \succ (x', y, z)$$

Moreover, comeasurability implies coseparability according to Qin and Rommeswinkel (2018) which becomes obvious by observing that $(x, y, z) \sim_2 (\bar{x}, \bar{y}, \bar{z})$ implies in the case $n = 4$ and $i = 2$:

$$(1 - z)(0, \frac{x_2}{1 - z}, 0, \frac{x_4}{1 - z}) \oplus z(\frac{x_1}{z}, 0, \frac{x_3}{z}, 0)$$

$$\sim (1 - z)(0, \frac{\bar{x}_2}{1 - z}, 0, \frac{\bar{x}_4}{1 - z}) \oplus z(\frac{\bar{x}_1}{z}, 0, \frac{\bar{x}_3}{z}, 0)$$

and thus Qin and Rommeswinkel (2018)’s coseparability of $X$ and $Y$ given $Z$ is implied by comeasurability. However, note that comeasurability is slightly stronger since it does not require the support of $x$ and $\bar{x}$ to be identical.

Thus, $\succ_i$ fulfills in the notation of Qin and Rommeswinkel (2018) $\hat{X} \perp \hat{Y} \mid \hat{Z}$. From pairwise strong essentiality of $\succ_i$ of at least four dimensions also follows that $\succ_i$ is such that dimensions $X$ and $Y$ are strictly essential if both $X$ and $Y$ each contain two of the pairwise strongly essential dimensions. What is left to show for the application of the quasi-separable representation theorem is that the well-behavedness conditions (Definition 1 in Qin & Rommeswinkel, 2018). The conditions i) (connectedness and openness of $\hat{S}$ in the product topology) and ii) (connectedness of dimensions) are fulfilled as can be easily verified. Condition iii) is not necessary by Remark 1 of Qin and Rommeswinkel (2018); if holding fixed $z \in Z$, the resulting space is a product space, then indifference curves need not be connected. From Theorem 1 of Qin and Rommeswinkel (2018), we then have the following representation result for all $i < n$ such that
$(i, n)$ are pairwise essential:

\[
(x_1, \ldots, x_n) \succsim (x_1', \ldots, x_n')
\]
\[
\iff (x, y, z) \succsim_i (x', y', z)
\]
\[
\iff f_i(x, z) + g_i(y, z) \geq f_i(x', z') + g_i(y', z')
\]
\[
\iff f_i(x, \sum_{k \neq i, n} x_k) + g_i((x_k)_{k \neq i, n-1, n}, \sum_{k \neq i, n} x_k) \geq
\]
\[
f_i(x, \sum_{k \neq i, n} x_k) + g_i((x_k)_{k \neq i, n-1, n}, \sum_{k \neq i, n} x_k)
\]
\[
\iff U_i(x_1, \ldots, x_n) \geq U_i(x_1', \ldots, x_n')
\]

and

\[
U_i(x) = f_i(x, \sum_{k \neq i, n} x_k) + g_i((x_k)_{k \neq i, n-1, n}, \sum_{k \neq i, n} x_k)
\]
\[
= f_i(x, \sum_{k \neq i, n} x_k) + g_i((x_k)_{k \neq i, n}).
\]

Using comeasurability, we can ensure during the utility construction process of $U_i, U_j$ that

\[
U_i(x) = U_j(x) = U(x).
\]

for all $i, j < n$. To see this in the four dimensional case, starting from any point $(x_1, x_2, x_3, x_4)$, we have that if

\[
(x_1', x_2, x_3, x_4') \sim (x_1, x_2', x_3, x_4')
\]
\[
(x_1', x_2', x_3', x_4) \sim (x_1'', x_2, x_3, x_4)
\]

then by comeasurability

\[
(x_1', x_2', x_3', x_4) \sim (x_1'', x_2, x_3', x_4').
\]

and it follows that for any utility interval from $U_1(x)$ to $U_1(x') = U_1(x'')$

\[
\frac{U_1(x_1', x_2, x_3, x_4') - U_1(x_1, x_2, x_3, x_4)}{U_1(x_1', x_2', x_3, x_4) - U_1(x_1, x_2, x_3, x_4)} = \frac{U_2(x_1, x_2', x_3, x_4') - U_2(x_1, x_2, x_3, x_4)}{U_2(x_1', x_2, x_3', x_4) - U_2(x_1', x_2, x_3', x_4)}
\]

and thus $U_1$ and $U_2$ are identical up to a suitably chosen affine transformation. The generalization of the previous arguments to higher dimensions is, though notationally cumbersome, straightforward.
Thereof without loss of generality,
\[ f_i(x_i, \sum_{m \neq i,n} x_m) + g_i((x_m)_{m \neq i,n}) = f_j(x_j, \sum_{m \neq j,n} x_m) + g_j((x_m)_{m \neq j,n}). \]

Setting \( x_m = \epsilon \) for all \( m \neq i, j, k \), we obtain:
\[ f_i(x_i, x_j + x_k) + g_i(\epsilon, \ldots, \epsilon, x_j, x_k) = f_j(x_j, x_i + x_k) + g_j(\epsilon, \ldots, \epsilon, x_i, x_k). \]

By Lemma 1, this functional equation has the solution \( f_i(x_i, x_j + x_k) = u_i(x_i) + \tilde{u}_i(x_i + x_j + x_k) + \tilde{u}_i(x_j + x_k) \) and \( f_j(x_j, x_i + x_k) = u_j(x_j) + \tilde{u}_j(x_i + x_j + x_k) + \tilde{u}_j(x_i + x_k) \). We therefore have for all \( i \),
\[
U(x) = u_i(x_i) + \tilde{u}_i(\sum_{m \neq n} x_m) + \hat{g}_i((x_m)_{m \neq i,n}) = u_j(x_j) + \tilde{u}_j(\sum_{m \neq n} x_m) + \hat{g}_j((x_m)_{m \neq j,n})
\]
with \( \hat{g}_i((x_m)_{m \neq i,n}) = g_i((x_m)_{m \neq i,n}) - \tilde{u}(\sum_{m \neq i,n} x_m) \) and \( \hat{g}_j((x_m)_{m \neq j,n}) = g_j((x_m)_{m \neq j,n}) - \tilde{u}(\sum_{m \neq j,n} x_m) \). Setting \( u_n(x_n) = \tilde{u}(1 - x_n) \) we obtain:
\[ U(x) = u_i(x_i) + u_n(x_n) + \hat{g}_i((x_m)_{m \neq i,n}). \]

Our initial choice of \( n \) was arbitrary. We have thus shown that for any \( i, n \), the utility representation is additively separable in strongly essential dimensions. For a dimension \( k \) that is not strongly essential, observe that either the dimension is never essential when combined with any other dimension at any amount allocated to these dimensions, or it is strongly essential for some localized subset of \( P \). We can then repeat the above arguments for the localized subset of \( P \) to obtain additive separability of the dimension \( k \) as well. Thus,
\[ U(x) = \sum_{i=1}^{n} u_i(x_i). \]

Uniqueness of the representation up to affine transformations follows from the uniqueness properties of the quasi-separable representations \( U_i \). Finally, we can continuously extend the real valued representation to the closure of the simplex except points where for some \( i, x_i = 1 \), since it is possible that \( \lim_{x_i \to 1} u_i(x_i) = \infty \). By continuity, all points with infinite utility are indifferent. \( \square \)
References


