Measuring Freedom in Games

Hendrik Rommeswinkel*

December 10, 2019

Abstract

The paper provides an analogue to Harsanyi's impartial observer theorem for game forms. Behind the veil of ignorance, a policy maker ranks combinations of game forms and information about how players interact within the game forms. The paper presents axioms on the preferences of the policy maker that are necessary and sufficient for the policy maker's preferences to be represented by the sum of an expected valuation and a freedom measure. The freedom measure is the mutual information between players' strategies and the individual outcomes of the game, capturing the degree to which players control their outcomes. The measure generalizes several measures from the opportunity set based freedom literature to situations where agents interact. This allows freedom to be measured in general economic models and thus derive policy recommendations based on the freedom instead of the welfare of agents. To illustrate the measure and axioms, applications to civil liberties and optimal taxation are provided.

Keywords: Freedom of Choice, Mutual Information, Entropy, Measurement, Game Form, Process, Income Taxation, Civil Liberties, Impartial Observer Theorem

JEL Classification: D63, D71, D81

---

*National Taiwan University, No. 1, Sec. 4, Roosevelt Rd., Taipei 106, rommeswi@ntu.edu.tw. I thank presentation participants at the Public Choice Society Meeting, Royal Economic Society Meeting, European Economic Association Conference, Stony Brook Game Theory Conference, LSE Choice Group, California Institute of Technology, Carlos III University of Madrid, Erasmus University Rotterdam, University of Groningen, Hamburg University, Hitotsubashi University, Karlsruhe Institute of Technology, Monash University, University of St. Gallen, and National Taiwan University for their helpful comments. Special thanks go to Martin van Hees, Chris Hitchcock, Martin Kolmar, Philip Pettit, Clemens Puppe, and Peter Wakker for more detailed comments on earlier versions of this paper. Financial support by the Swiss National Science Foundation through grants P1SGP1_148727 and P2SGP1_155407 and by the Taiwan Ministry of Science and Technology through grants 107-2410-H-002-031 and 108-2410-H-002-062 is gratefully acknowledged.

1 Introduction

Normative judgments are commonly made in economics using one of two gold standards. The first is welfarism, judging states of society by Pareto improvements. The second is utilitarianism, judging states of society by aggregate expected utility. The two criteria lie on opposite ends of a spectrum; utilitarianism achieves a complete (cardinal) ordering but relies on utility information that is not directly accessible to the economist. Welfarism compromises on the existence of a complete ordering in favor of a criterion that solely relies on more accessible information about ordinal preferences.

In favor of utilitarianism, Harsanyi's impartial observer theorem (Harsanyi, 1953b, 1955, 1977) provides an answer to how a policy maker should aggregate individual utilities into a welfare criterion.[1] Under the assumptions of rationality of the policy maker, rationality of the affected individuals, and the acceptance principle, the policy maker evaluates lotteries according to the weighted sum of the expected utility of the individuals. However, policy makers may still feel discomfort with applying utilitarian criteria to policy evaluation.[2] Possible reasons are firstly the failure of utilitarianism to account for procedural considerations, secondly the difficulty of measuring interpersonally comparable utility functions from observational data, and thirdly Harsanyi's requirement that individuals are rational expected utility maximizers.

In this paper, we provide an impartial observer theorem of institutional choice that addresses these problems. The first problem is addressed by modeling institutions as game forms. By allowing for interactions between players, the policy maker can have procedural preferences, e.g., preferences over who can influence what outcome. The second problem is addressed by forcing the policy maker to rank institutions based only on the game form and the policy maker's information about behavior in the game form. This information is represented as probability measures on the players' chosen strategies. Thirdly, no impositions are made with respect to equilibrium concepts or the rationality of players in the game. Instead, as a result of the axiomatization, the policy maker attaches either positive or negative value to players controlling their outcomes.

[1]Generalizations of the impartial observer theorem are provided by Karni and Weymark (1998), Safra and Weissengrin (2003), Gilboa, Samet, and Schmeidler (2004), Gajdos and Kandil (2008), and Grant, Kajii, Polak, and Safra (2010). See Weymark (2011) for an excellent discussion of Harsanyi's theorems and critiques thereof.

[2]For a discussion of the limits of utilitarianism as a normative criterion for public policy, see for example Pattanaik (2009).

We call a game form endowed with the policy maker's information about how players will interact a *process*. The policy maker forms preferences over processes in compliance with the following axioms. The Rationality axiom imposes completeness and transitivity of the preference relation. Continuity and Outcome Equivalence ensure that similar processes are similarly ranked. Lottery Independence requires the policy maker to obey the von Neumann-Morgenstern independence axiom for pure lotteries over outcomes.

The central axioms are Strategy Independence and Subprocess Monotonicity. Strategy Independence deals with situations in which the policy maker learns that a choice between strategies was actually made by nature. Thus, instead of a player making a choice between strategies, instead nature randomly chooses the strategy for the player. The axiom requires that ceteris paribus, the change of value due to this choice removal is independent of the other choices being made. For example, if the policy maker learns that aversion to bitter vegetables is determined genetically (Wooding et al., 2004), then the resulting change in the policy maker's preference is independent of the policy maker's preference change resulting from learning that smoking behavior is genetically determined (Erzurumluoglu et al., 2019). Precisely, the policy maker may not prefer that vegetable choices are determined by nature if and only if smoking choices are also determined by nature. Instead, the policy maker needs to make independent judgments about the desirability of the agent (rather than nature) being in control of strategic choices.

Subprocess Monotonicity requires the policy maker's preference for a process to be increasing in the preference of its subprocesses. A subprocess is the process obtained from conditioning the probability measures on behavior to a subgame. Monotonicity in the value of a subprocess is only required to hold when the subprocess reaches distinct outcomes from the remainder of the game. Consider as a simple example the process in which a single player gets to choose between smoking and not smoking. According to the information of the policy maker, both smoking and not smoking are equally likely to be chosen. This process has two trivial subprocesses, one in which the player smokes with certainty and one in which the player does not smoke with certainty. Suppose the policy maker prefers to dictatorially assign not smoking to dictatorially assigning the player to smoke. Then subprocess monotonicity without the requirement of disjoint outcomes would imply that the substitution of the subprocess in which the player smokes with certainty by a subprocess in which the player does not smoke with certainty would improve the process. However, the resulting process would be the trivial choice between not smoking and not smoking. When we substitute a subprocess by another subprocess it may therefore occur that meaningful choices are removed if some of the outcomes of the subprocess overlap with outcomes of the remainder of the process. Therefore, we require Subprocess Monotonicity only to hold if the outcomes of the subprocess are disjoint from the remainder of the game.

We obtain a representation theorem according to which the policy maker's preferences are additively separable across players. For each player, the policy maker's evaluation of the process consists of the sum of two components. The first component is an expectation across the valuation of individual outcomes that can be interpreted as the policy maker's perceived instrumental value of the process for the player.[3] The second component is the mutual information between the player's strategies and outcomes. This component is interpreted as a freedom measure; it measures the degree to which players control their outcomes. Under the special case of perfect control, the mutual information becomes equal to the Shannon entropy of the outcomes, a freedom of choice measure suggested by Suppes (1996).

The present paper therefore brings two literatures together, the literature on impartial observer theorems and the freedom of choice literature. The contribution to the impartial observer theorem literature is the conclusion that if the impartial observer chooses between institutions in which players have agency and the policy maker has only information about behavior, then the policy maker should employ a freedom of choice measure to evaluate these institutions. Under the assumption that behavior is a sufficient statistic for utilities, a utilitarian policy maker would need to justify how any deviation from the axioms imposed in this paper would help approximate individual utilities from behavior. The contribution to the freedom of choice literature is a solution to the problem posed in Pattanaik (1994). Pattanaik (1994) showed that opportunity set based measures of freedom of choice encounter problems when being applied to situations in which agents interact. The difficulty arises because in situations in which agents interact, opportunity sets from which agents can freely choose are no longer clearly defined. The choice of one agent may influence the available opportunities of another agent and vice versa. This problem has prevented the literature to provide measures even for a simple exchange economy as Pattanaik (1994) showed. Yet, it is exactly these cases when agents depend on each other

[3]Since the policy maker has no information about player's utilities beyond the behavior, this expectation is the policy maker's subjective evaluation of how desirable the outcomes are, not the player's.

to achieve their goals, when they exhibit power over each other, or when they are coerced by others that the measurement of freedom becomes interesting. The lack of freedom measures for situations where agents interact therefore creates an undesirable wedge between the normative analysis that can be performed by economists and normative perceptions outside economics.

To show that the measure axiomatized in this paper effectively solves the problem of measuring freedom when agents interact, we apply the measure to two examples. The first example is a simplified model of racial discrimination on buses in Montgomery in the early 1950s. We analyze a game form representing the interaction between a passenger and a driver. Using historical accounts, we can inform a policy maker about how the passengers and drivers interacted. According to the law, no passenger had to yield their seat to another passenger. However, black passengers were frequently required to yield their seats for white passengers. In case they refused to yield their seat, they were arrested and economically sanctioned. We show how this discrimination leads to a reduction of freedom of choice. The example also shows why we include the policy maker's information about the strategies of the players; neither a game form in which only the *legal* actions are included, nor a game form in which all *possible* actions are included would correctly capture the degree of freedom of choice of the players.

The second example is a production economy, a similar problem to the one posed by Pattanaik (1994). A production economy can be treated as a process in which the consumers choose demand functions as fully contingent plans that yield a final outcome, the allocation. In our model, the policy maker has information about the reported demands of all players and the production conditions. The information of the policy maker behind the veil of ignorance is such that the players choose their demand functions in a non-strategic manner which is the central property of competitive equilibria. Using the model of a utility maximizing consumer, the uncertainty about demand can be translated into uncertainty about a preference parameter without assuming the policy maker's knowledge of cardinally comparable utility functions. Further, the policy maker is uncertain about consumers' individual productivity and aggregate shocks. Freedom is measured as the degree to which preferences determine consumption and labor outcomes. The limitations to freedom are given by individual's variations in ability and shocks that reduce the causal connection between demand functions and consumption outcomes. In this model, we analyze how a policy maker optimally sets the tax progressivity to maximize freedom.

The paper continues as follows. Section 2 reviews the literature of freedom of choice measures, with a focus on the ones related to the measure developed here. Section 3 begins with an informal description of the policy maker's problem and then provides the game theoretic framework in which the measure is developed. Section 4 axiomatizes the measure. The application of the freedom measure to a production economy and the problem of optimal income tax progression is given in Section 5.

2 Freedom Measures

Philosophers and economists alike have stressed the intrinsic importance of freedom (e.g., Berlin, 1958; Sen, 1988). To this end, the freedom of choice literature,[4] following the seminal contributions of Pattanaik and Xu (1990) and Jones and Sugden (1982), provides measures that can be used to determine the freedom offered by an opportunity set. In the following, the measures most closely related to the measure proposed in this paper will be reviewed. All measures will be indexed by the authors' last names. We begin with measures based on opportunity sets. A freedom relation $\succsim_F$ holds between subsets $\mathcal{C}$ of $\mathcal{X}$. $\mathcal{C} \succsim_F \mathcal{C}'$ with $\mathcal{C}, \mathcal{C}' \subseteq \mathcal{X}$ can be interpreted as 'the opportunity set $\mathcal{C}$ offers weakly more freedom than the opportunity set $\mathcal{C}'$'. The measure of Pattanaik and Xu (1990) states that the freedom offered by an opportunity set $\mathcal{C}$ is its cardinality $|\mathcal{C}|$, that is:

Definition 1. Cardinality Measure (Pattanaik and Xu, 1990)
Suppose $\mathcal{C}, \mathcal{C}' \subseteq \mathcal{X}$. Then $\mathcal{C} \succsim_{F,PX} \mathcal{C}' \Leftrightarrow |\mathcal{C}| \geq |\mathcal{C}'|$.

A possible issue of this measure is that it may count alternatives that no reasonable agent would ever choose. Jones and Sugden (1982) proposed a measure based on a set of so-called "reasonable" preference relations $\mathcal{R}$ and freedom is measured according to the set of reasonably chosen alternatives $\{x \in \mathcal{C} : \exists R \in \mathcal{R} : \forall y : xRy\}$. While the precise definition of "reasonable" is left open, Jones and Sugden (1982) give as an example the choice of a prisoner, who can either "stay in the cell" or "get shot". Since it would be unreasonable to prefer getting shot to staying in the cell, the set of reasonably chosen alternatives is the singleton "stay in the cell". On the basis of the ideas developed by Jones and Sugden (1982), Pattanaik and Xu (1998) axiomatize the following measure:

[4]For surveys of the literature, see Barberà, Hammond, and Seidl (2004), Baujard (2007), or Dowding and van Hees (2009).

Definition 2. Reasonable Preference Measure (Jones and Sugden, 1982; Pattanaik and Xu, 1998)
Suppose $\mathcal{C}, \mathcal{C}' \subseteq \mathcal{X}$. Then $\mathcal{C} \succsim_{F,JS} \mathcal{C}'$ iff

$$|\{x \in \mathcal{C} : \exists R \in \mathcal{R} : \forall y : xRy\}| \geq |\{x \in \mathcal{C}' : \exists R \in \mathcal{R} : \forall y : xRy\}|.$$

The measure thus states that the freedom an opportunity set offers can be measured by the cardinality of the set of reasonably chosen alternatives.

It has been argued that freedom of choice is strongly connected to diversity. Individuals are more free if they are able to make choices over a more diverse opportunity set. Two types of diversity have been identified: Qualitative diversity refers to how distinct elements of a set are and has been given a formalization in Nehring and Puppe (2008). Quantitative diversity refers to the relative frequencies with which different objects are chosen and can for example be measured by the Shannon (1948) entropy. Suppes (1996) proposes to measure freedom as the entropy of the relative frequencies with which an agent chooses the alternatives of an opportunity set:

Definition 3. Entropy Freedom Measure (Suppes, 1996)
$F_S(\mathcal{C}, P) = -\sum_{x \in \mathcal{C}} P(x) \ln P(x)$ where $P(x)$ refers to the probability with which an agent chooses element x of the opportunity set $\mathcal{C}$.

The entropy measure increases both in the total number of options chosen with positive probability and how even the distribution of these chosen outcomes is. The entropy freedom measure can be seen as a generalization of the reasonable preference measure if the distribution P is interpreted as a "degree of reasonability" since the entropy is increasing in the total number of evenly distributed elements.

The closest in spirit to our model is the literature on freedom of choice in game forms (Peleg, 1997; Braham, 2006; Bervoets, 2007; Ahlert, 2010). Moving from opportunity sets to a more general framework was an important conceptual innovation. This moved the quest for a proper measure of freedom from measuring numbers of alternatives to measuring control over choice.[5] This is important since cases of actual policy relevance (discrimination, consumer freedom, political participation) are unlikely purely decision theoretic; difficult policy tradeoffs commonly involve the freedoms of multiple individuals.

Bervoets (2007) suggests using the maxmin criterion to rank game forms according to which the game form is ranked by the best element

[5]For a classification of the richness of possible limitations to freedom by other individuals, see Carter (2013).

a player can guarantee to obtain. Ahlert (2008) ranks game forms according to the maxmin criterion on the guaranteed level of well-being provided by a society. Ahlert (2010) phrases the question of a measure of freedom of choice from the perspective of a policy maker (an approach which we follow here). In this framework, Ahlert (2010) measures the sets of alternatives that can be determined and the sets of alternatives that can be excluded via a lexicographic cardinality rule. The central difference of the current paper to the work on ranking game forms is that our policy maker ranks game forms combined with probabilistic information about players' behavior. For a stylized example of why this is important, consider a game in which a player may choose between actions that deteriorate democratic instutions or improve these institutions. This decision power of the player may be evaluated very differently depending on the policy maker's information about whether the player will indeed take this action. Therefore, the ranking leaves out important information in case it relies only on the information contained in the game form.

A precursor to the idea of measuring freedom via the (probabilistic) degree to which individuals control their outcomes can be found in the measure by Braham (2006) which endows game forms with probabilities to account for interactions between agents. The measure captures the degree to which an individual i can force a certain outcome x to come about in the game. With some abuse of notation the measure states:

Definition 4. Game Form Measure (Braham, 2006)
$F_B(x, i) = P(\text{outcome is } x | i \text{ chooses } x) = \frac{P(\text{outcome is } x \text{ and } i \text{ chooses } x)}{P(i \text{ chooses } x)}$

where it may occur that $P(\text{outcome is } x | i \text{ chooses } x) < 1$ because the actions of the other agents may lead to another outcome, even if i chooses x. The measure therefore takes up the idea that an agent is free if he can force certain outcomes to occur. Unlike the measure by Braham (2006), the measure in the present paper employs probability measures over agent's strategies and accounts for multiple outcomes.

We briefly give an overview over the remaining literature. Similar to the game form approach, Bossert (1998) and Arlegi and Dimitrov (2009) treat the options offered by opportunity sets as only indicative of the outcomes the individual can achieve. Qualitative diversity and characteristics of opportunity sets are analyzed in Rosenbaum (2000), van Hees (2004), Nehring and Puppe (2008, 2009). The qualitative diversity of sets of lotteries is measured in Gustafsson (2010), Sher (2018). Puppe and Xu (2010) and Ryan (2016) add information about essential elements to opportunity sets. An extension of the opportunity set based approach is to also include information on the constraint from which the options are chosen (Bavetta & del Seta, 2001). Unstable preferences as a source of preference for flexibility/freedom are considered in (Koopmans, 1964; Kreps, 1979; Sugden, 2007). The idea of multiple preference relations as in Jones and Sugden (1982) has been further examined by Sugden (1998), Nehring and Puppe (1999), and Bavetta and Peragine (2006). An important topic is also the distribution of freedom between individuals, for which a survey is given by Peragine (1999). Broader discussions are given by Carter (1992), Carter (1995), van Hees and Wissenburg (1999) Bavetta (2004), Carter (2004), Kolm (2010), and Shnayderman (2016).

3 The Model

A policy maker faces a decision problem in which she decides behind a veil of ignorance between establishing different institutions. The institutions are modeled as game forms between a set of players over lotteries of *social outcomes*. The policy maker has information about how players will interact in the game form. More precisely, the policy maker is given a probability measure over the (possibly mixed) strategies of each player. The combination of a game form with the policy maker's information about strategies is called a *process*. Processes can differ on the game form, contain identical game forms but different information about strategies, or differ in both respects. The policy maker forms preferences over processes. We impose axioms on the preferences of the policy maker and prove a representation theorem. The representation of the policy maker's preferences consists of a sum across players of the sum of an expectation across outcome and a measure of how informative strategies are about outcomes. We interpret the first component of the sum as an instrumental value of the process; it measures the degree to which the policy maker believes the individuals obtain desirable outcomes. The second component of the sum is interpreted as a freedom measure, it measures the extent to which players exercise control over their outcomes and how many outcomes they control.

The central idea of the axiomatization is that nature-driven uncertainty is valued differently from agent-driven uncertainty by the policy maker. For pure lotteries in which no player is influential, the policy maker obeys the classical independence axiom. When players are influential, uncertainty about their strategies is not commensurable with uncertainty from lotteries. The reason is that strategies allow players to express their individual likes and dislikes for various outcomes. For example, the policy maker may find it more compelling that the colors of a players bedroom walls depend on strategic uncertainty about that player's behavior and not the mixed strategy of another player or a lottery. To account for this, each *social outcome* is a combination of *individual outcomes* of the players.

The decision problem resembles that of an impartial observer theorem (Harsanyi, 1953a, 1977, 1955). The framework used in this paper distinguishes itself in several aspects, however. The impartial observer theorem makes use of extended lotteries in which the decision maker faces uncertainty about her identity in society and uncertainty about which outcome is implemented. Our policy maker instead faces uncertainty about how the individuals in society interact with another. Allowing for interactions between players allows us to account for concerns of procedural fairness. The impartial observer theorem requires the policy maker to know the expected utility functions of the different positions in society. In practice, the information problem that needs to be solved to implement utilitarianism is infeasible; we would need to not only derive the expected utility functions for every member of society but also find the proper scale for cardinal comparisons.[6] In place of utility information, our policy maker only needs to acquire information about the strategic choices of the players in the game. This information can in principle be given by a dataset on the behavior of players or by a theoretical model. Realistically obtainable information about behavior therefore permeates our policy maker's veil of ignorance while inaccessible utility information does not. Finally, the impartial observer theorem makes assumptions about the rationality of preferences not only of the policy maker but also of all individuals affected by the policy. While our framework imposes rationality conditions on the policy maker, it does not impose these on the players of the game form.

3.1 Notation

f, g, h denote generic functions $f : x \mapsto f[x]$. $f|_{\mathcal{Z}}$ denotes the restriction of $f : \mathcal{X} \to \mathcal{Y}$ to the subset $\mathcal{Z} \subseteq \mathcal{X}$ of the domain. If $f : \mathcal{X} \to \mathcal{Y}$, then $f[\mathcal{Z}] = \{y \in \mathcal{Y} : \exists z \in \mathcal{Z} : y = f[z]\}$ is the image of the function of the

[6] The reader may imagine having estimated that two individuals have distinct constant relative risk aversion utility functions over money. At which monetary level should the marginal utility of income be identical? The problem is worsened in case one gives up the commensurability of outcome and identity lotteries. In this case, the utilitarian must find the correct continuous monotone transformations of the expected utility representations, (Grant et al., 2010).

set $\mathcal{Z} \subseteq \mathcal{X}$. When a set $\mathcal{X}$ is understood as a subset of $\mathcal{Y}$, then $\mathcal{X}^C = \mathcal{Y} \backslash \mathcal{X}$ denotes the complement.

If $\mathcal{S}$ is a topological space, then $\Delta\mathcal{S}$ denotes the finite support probability measures over the Borel sigma algebra of $\mathcal{S}$. The support of $\nu \in \Delta\mathcal{S}$ is denoted by $supp[\nu]$. In case of a finite set $\mathcal{S}$, we assume the discrete topology and therefore $\nu \in \Delta\mathcal{S}$ means the domain of ν is the power set $2^{\mathcal{S}}$. We will frequently simplify notation by writing $\nu[s]$ instead of $\nu[\{s\}]$ for singletons. If f is a measurable mapping from $\mathcal{S}$ to $\mathcal{S}'$, then $f\#\mu$ is the pushforward measure fulfilling $f\#\mu[s] = \mu[f^{-1}[s]]$ for all $s \in \mathcal{S}'$. If $s \in \mathcal{S}$, then $\mathbb{1}_s \in \Delta\mathcal{S}$ fulfills $\mathbb{1}_s[s] = 1$. If $\nu \in \Delta\mathcal{S}$, $\mathcal{S}' \subset \mathcal{S}$, and $\nu[\mathcal{S}'] > 0$, then the conditional probability measure denoted by $\nu|\mathcal{S}'$ fulfills $(\nu|\mathcal{S}')\,[\mathcal{S}''] \cdot \nu[\mathcal{S}'] = \nu[\mathcal{S}'']$ for $\mathcal{S}'' \subseteq \mathcal{S}'$.

For any two probability measures $\nu \in \Delta\mathcal{S}, \nu' \in \Delta\mathcal{S}'$, we can assign a product measure $\nu \otimes \nu' \in \Delta(\mathcal{S} \times \mathcal{S}')$, such that $(\nu \otimes \nu')[s, s'] = \nu[s]\nu'[s']$. For finitely many products of a set of measures, $\mathcal{D} = \{\nu_1, ..., \nu_n\}$, we can write $\bigotimes_{\nu \in \mathcal{D}} \nu = \nu_1 \otimes ... \otimes \nu_n$.

A space of probability measures is a set endowed with the product topology. Therefore, it is meaningful to write $\Delta\Delta\mathcal{S}$ as the space of finite support probability measures over finite support probability measures over $\mathcal{S}$.

For any two probability measures over the same set $\nu, \nu' \in \Delta\mathcal{S}$, we can define the mixture of the two probability measures $\alpha\nu \oplus (1-\alpha)\nu' \in \Delta\mathcal{S}$ as the probability measure that fulfills for all $s \in \mathcal{S}$: $\alpha\nu \oplus (1-\alpha)\nu'[s] = \alpha\nu[s] + (1-\alpha)\nu'[s]$. For a probability measure $\alpha \in \Delta\mathcal{S}'$ and a one-to-one function $f : \mathcal{S}' \to \Delta\mathcal{S}$, we define

$$\bigoplus_{s'} \alpha[s']f[s'] = \alpha[s'_1]f[s'_1] \oplus (1 - \alpha[s'_1]) \left(\frac{\alpha[s'_2]}{1 - \alpha[s'_1]} f[s'_2] \oplus \dots \right) \quad (1)$$

In other words, f and α can together be interpreted as a two stage probability measure over $\mathcal{S}$ and the measure $\bigoplus_{s'} \alpha[s']f[s']$ is its reduction to a single stage.

3.2 Game Forms

Let $\mathcal{N}$ be a set of players. We assume there exists some universal set of social outcomes $\mathcal{O}$. Outcomes are denoted by lowercase letters from the end of the alphabet, x, y, z. For each player i, there exists a set $\mathcal{O}_i$ of individual outcomes $x_i, \dots$ that are a partition of $\mathcal{O}$. If the policy maker is of the opinion that the difference in outcomes x and y are irrelevant[7] for player i, then the individual outcome of player i is the same in both outcomes, i.e., $\exists x_i \in \mathcal{O}_i : x, y \in x_i$. For simplicity, we assume that all combinations of individual outcomes are possible, i.e., $\bigcap_{i \in \mathcal{N}} x_i \subseteq \mathcal{O}$.

The set of game forms with possible outcomes $\mathcal{O}$ is denoted by $\mathcal{G}[\mathcal{O}]$. We define strategic game forms as follows.

Definition 5 (Strategic Game Form)**.** A strategic game form $G \in \mathcal{G}[\mathcal{O}]$ is a tuple $(\mathcal{N}, \mathcal{A}, o)$ where

— $\mathcal{N} = \{1, ..., n\}$ is a finite set of players.

— $\mathcal{A} = \prod_{i \in \mathcal{N}} A_i$ is the set of action profiles.

— $\mathcal{A}_i$ is the finite set of actions of player i.

— $o : \mathcal{A} \to \Delta\mathcal{O}$ is the outcome function specifying for each action profile a lottery.

Lowercase letters from the beginning of the alphabet $a, b, c, \ldots \in \mathcal{A}$ always denote action profiles, action profiles with a subsript, $a_i \in A_i$ denote an action taken by player i. To avoid the awkward notation $(a_1, ..., a_{i-1}, a_i, a_{i+1}, ..., a_n)$ where $a_j \in \mathcal{A}_j$, we employ the notation (a_i, a_{-i}) for such tuples. Each action profile results in a lottery over outcomes. In the following, G always denotes an arbitrary strategic game form with the same set of players $\mathcal{N}$.

Example. To clarify the various concepts used and defined in this paper, we employ an example of discrimination. The game form[8] is shown in Table 1. In it, player 1, decides how to get to work. Each action is a fully contingent plan. She can choose to walk, a_1^5, in which case she arrives with certainty at work, outcome z. Alternatively, she can attempt to take the bus, $a_1^1, \ldots, a_1^4$. The driver player 2, decides whether to reject her as a passenger, a_2^3, in which case player 1's choice of strategies between $a_1^1, \ldots, a_1^4$ determines whether she walks and arrives delayed at work, x, or cancels the journey, y. If player 1 does not get rejected by the driver, she is requested to give up her seat to a white passenger. With actions

[7] Ahlert (2010) instead employs a perception function that distinguishes social states according to whether individuals perceive the states to be different. Instead of making this a question of perception, we make this a normative issue to be determined by the policy maker.

[8] A similar game form is given in Mailath, Samuelson, and Swinkels (1993). The interpretation of the game form is a much simplified account of the segregation laws and discriminatory practices of bus lines in Montgomery, Alabama up to 1956. For historical accounts, see Phibbs (2009), Burns (2012), Theoharis (2015).

a_1^1 and a_1^2 she yields her seat to the white passenger and stands for the remainder of the bus ride, u. With the actions a_1^3 and a_1^4 she insists on her right to sit. In case the driver acts lawfully, a_2^1, player 1 gets to sit during the bus ride, v. In case the driver calls the police, a_2^2, player 1 gets arrested and loses her job, w. We note that in this game there is no uncertainty

	lawful a_2^1	police a_2^2	reject a_2^3
bus, yield, walk a_1^1	u	u	x
bus, yield, cancel a_1^2	u	u	y
bus, refuse, walk a_1^3	v	w	x
bus, refuse, cancel a_1^4	v	w	y
walk a_1^5	z	z	z

Table 1: Montgomery Bus Game Form

about the outcomes after the players' moves. Therefore, for example $o[a_1^1, a_2^1] = \mathbb{1}_u$. We must also determine what the relevant distinctions in outcomes are. Based on the above interpretation, we may for example be concerned about the freedom of choice of player 1 about whether and how she commutes to work. This means that u, v, etc.., will generally be considered normatively distinct outcomes for player 1 by the policy maker and therefore belong to different elements of the partition $\mathcal{O}_1$. Some policy maker perhaps considers the delay from being rejected from riding the bus negligible and considers the outcomes z and x equivalent for player 1. Another policy maker may find that the driver's outcomes are all identical $\mathcal{O}_2 = \{\{u, v, w, x, y, z\}\}$. In any case, these normatively imposed distinctions must be made explicit.[9] *End of example.*

In some game forms, duplicate actions may exist. Generally, adding or removing such actions should have no effect on the interaction represented by the game form. We therefore require a notion of strategically equivalent actions.

Definition 6 (Strategically Equivalent Actions)**.** Two actions $a_i, a_i' \in \mathcal{A}_i$ are strategically equivalent given $\mathcal{B}_{-i} \subseteq \mathcal{A}_{-i}$, denoted $a_i \approx_{\mathcal{B}_{-i}} a_i'$, if for all $a_{-i} \in \mathcal{B}_{-i}$ we have that $o[a_i, a_{-i}] = o[a_i, a_{-i}]$. a_i, a_i' are strategically equivalent, $a_i \approx a_i'$, if they are strategically equivalent given $\mathcal{A}_{-i}$.

We now make definitions that allow us to remove all such duplicate actions from a game form. Denote by $\mathcal{A}_i/\approx$ the quotient set with respect

[9]As already argued by Sugden (2003), any measure of freedom of choice ultimately depends on the way the outcome space is partitioned. We will see in Section 5 how different partitions of the outcome space yield different policy objectives.

to the equivalence relation $\approx$, i.e., all actions $a_i \in \mathcal{A}_i$ are replaced by their equivalence class $\{a'_i \in \mathcal{A}_i : a_i \approx a'_i\}$. With some abuse of notation, denote by $\mathcal{A}/\approx = \prod_j \mathcal{A}_j/\approx$ the profiles resulting from replacement of all strategically equivalent actions by their equivalence class with canonical projection $f : a \mapsto \{a' \in \mathcal{A} : \forall i \in \mathcal{N} : a_i \approx a'_i\}$ from action profiles to their respective equivalence class.

Definition 7 (Reduced Form)**.** For a strategic game form $G = (\mathcal{N}, \mathcal{A}, o)$, the reduced strategic form $G/\approx$ is given by $(\mathcal{N}, \mathcal{A}/\approx, f \circ o)$.

Throughout the paper, we assume the game form is given by the reduced form.

Example. Consider the Montgomery bus game with the adjustment that $x = y$. In this case, the reduced form is given in Table 2. *End of example.*

	a_2^1	a_2^2	a_2^3
$\{a_1^1, a_1^2\}$	u	u	x=y
$\{a_1^3, a_1^4\}$	v	w	x=y
a_1^5	z	z	z

Table 2: Reduced Form Example

Information is an important aspect in strategic interactions between individuals. Commonly, information sets are defined in extensive form games. Mailath et al. (1993) show that information sets can also be defined in strategic game forms in a corresponding manner.

Definition 8 (Normal Form Information Set)**.** In a reduced normal form G, the set $\mathcal{B} \subseteq \mathcal{A}$ is a normal form information set of player i if

$$\mathcal{B} = \mathcal{B}_i \times \mathcal{B}_{-i} \tag{2}$$

$$\forall a_i, a'_i \in \mathcal{B}_i, \exists a''_i \in \mathcal{B}_i : \quad a_i \approx_{\mathcal{B}_{-i}} a''_i, \quad a'_i \approx_{\mathcal{A}_{-i} \setminus \mathcal{B}_{-i}} a''_i. \tag{3}$$

It follows immediately that every profile $a \in \mathcal{A}$, is an information set for every player. If a set of actions is an information set for every player, then it is a subgame:[10]

Definition 9 (Normal Form Subgame)**.** $\mathcal{B} \subseteq \mathcal{A}$ is a normal form subgame if it is a normal form information set for each player.

[10]Formally, $\mathcal{B}$ by itself is not a game form. However, together with the original game form G, $\mathcal{B}$ uniquely determines the game form $(\mathcal{N}, \mathcal{B}, o|_{\mathcal{B}})$. We follow Mailath et al. (1993) in calling $\mathcal{B}$ a normal form subgame.

Example. In Table 1, the set $\{a_1^1, \ldots, a_1^4\} \times \{a_2^1, a_2^2\}$ is a subgame. We verify (3) of Definition 8 for the two actions a_1^1 and a_1^4. The condition requires that there is an action that agrees with a_1^1 on the subgame and with a_1^4 elsewhere. This action is given by a_1^2, as it is strategically equivalent with a_1^1 given $\{a_2^1, a_2^2\}$ and strategically equivalent with a_1^4 given $\{a_2^3\}$. Intuitively, this means that the action player 1 plays on the subgame can be independently chosen from the action outside the subgame. It is straightforward to check the remaining actions in a similiar manner. *End of example.*

In line with Mailath et al. (1993), Theorem 3, the relation between normal form subgames with extensive form subgames can be summarized as follows. For every normal form game G with a normal form subgame G', there exists an extensive form game E with a reduced normal form G that contains an (extensive form) subgame E' with a reduced normal form G'. Therefore, if the reduced normal form contains all strategically important features of an extensive form (Thompson, 1952; Elmes & Reny, 1994), then employing normal form subgames and normal form information sets is without loss of generality in the axiomatics. Since the policy maker also has information about the behavior in the game form, this does not commit us to assuming that individual's behavior is uninfluenced whether the game is perceived by players as an extensive form or in its reduced normal form. If the policy maker has information that the players perceive the game in a particular extensive form and act accordingly, then this should be reflected in the information about behavior.

We conclude this section with a summary of the concepts that have been introduced. We began by defining game forms in which after all players simultaneously chose an action, a lottery resolved which outcome occurred. Next, we defined strategic equivalence and the reduced form which equates all strategically equivalent actions. We furthermore used the notion that actions can be strategically equivalent with respect to some actions taken by other players to define information sets and subgames without making use of the extensive form.

3.3 Processes

The information contained in a game form is not always sufficient to make moral judgments. For example, a utilitarian policy maker would in addition like to know the utilities and the expected behavior of individuals. We will not permit utility information behind the veil of ignorance but information about player's behavior only. We define the strategies of the players as follows.

Definition 10 (Mixed Strategy)**.** A mixed strategy of player i, $\mu_i \in \Delta\mathcal{A}_i$ is a finite support probability measure over the actions, $\mathcal{A}_i$.

Definition 11 (Strategy Profile)**.** A strategy profile $\mu = \prod_{i\in\mathcal{N}} \mu_i$ specifies a strategy for each player.

Behind the veil of ignorance, the policy maker is uncertain about the choice of strategies of the player. This uncertainty is reflected in the behavior of the player. We therefore define the information of the policy maker as probabilistic beliefs.

Definition 12 (Information about Strategies)**.** The information of the policy maker about the strategies of player i, $\theta_i \in \Delta\Delta\mathcal{A}_i$ is a finite support probability measure over the strategies, $\Delta\mathcal{A}_i$. The policy maker's information about strategy profiles is given by $\theta = (\bigotimes_{i\in\mathcal{N}} \theta_i) \in \prod_{i\in\mathcal{N}} \Delta\Delta\mathcal{A}_i$.

It is noteworthy that the information about strategies exhibits independence across players. This implies that the description of the interaction provided by the game form is *comprehensive* in the following sense. According to the information of the policy maker, the players have no way of correlating their actions in any way that is not described by the game form. As we are interested in process value, this is important as the following example shows.

Example. Suppose in the Montgomery bus game form we observe that according to the information of the policy maker, player 1 plays the pure strategy a_1^5 if and only if player 2 plays a_2^2 and player 1 plays a_1^3 if and only if player 2 plays a_2^1. In this case, it seems that player 1 has perfect control over whether she walks, z, or sits on the bus, v. Similarly, player 2 seems to have perfect control over these two outcomes as well. This cannot be the whole story of the interaction between the two players, however. To coordinate their actions, they must either rely on a signal from nature (let's say weather) or one player is able to condition on the other's strategy. Consider the first scenario: on days with hot weather, player 1 walks and the driver is in a foul mood and would discriminate if player 1 tries to enter the bus. On days with cool weather, player 1 takes the bus and player 2 is in a good mood and does not discriminate. Alternatively, consider the second scenario: player 1 observes whether today's driver is known to discriminate and avoids taking the bus. If player 2 observes that today's driver is known not to discriminate, she takes the bus. The policy maker may want to judge the two situations differently. However, only knowing that the actions are correlated but not the reason for the correlation makes it impossible to determine which of the two scenarios is correct. To avoid this issue, the probability measure over strategy profiles is a product measure of the probability measures over individual strategies. *End of example.*

We now define the primitives over which the policy maker has preferences. These primitives determine what the policy maker sees behind the veil of ignorance. The policy maker forms preferences over processes from a set of processes defined as follows.

Definition 13 (Process Space)**.** The process space $\mathcal{P}$ is defined as

$$\mathcal{P} = \{(G, \theta) : G = (\mathcal{N}, \mathcal{A}, o) \in \mathcal{G}[\mathcal{O}], \theta \in \Delta\Delta\mathcal{A}\}. \tag{4}$$

Thus, processes are game forms endowed with information about strategies and the set $\mathcal{P}$ contains all possible processes given the set of outcomes $\mathcal{O}$.

Not all outcomes that are within a game form are reached with positive probability. While we have already introduced the support of measures, we now introduce the support of processes as the individual outcomes reached with positive probability.

Definition 14 (Outcome Support)**.** In a process (G, θ), the support of an action profile $a \in \mathcal{A}$ from the perspective of $i \in \mathcal{N}$ is defined as:

$$supp_{i,(G,\theta)}[a] = \{x_i \in \mathcal{O}_i : x_i \cap supp[o[a]] \neq \emptyset\} \tag{5}$$

We denote $supp_i[G, \theta] = supp_{i,(G,\theta)}[\bigcup_{\mu \in supp[\theta]} supp[\mu]]$ as the support of the process from the perspective of player i.

Example. Consider the outcomes that only differ by whether player 1 gets to sit during the bus ride, $\{v\}$, stand during the bus ride, $\{u\}$, gets arrested, w, or does not ride the bus, $\{x, y, z\}$. If the policy maker believes with certainty that player 1 plays either one of the actions a_1^1 and a_1^2 , then the support from the perspective of player 1 is $\{\{u\}, \{x, y, z\}\}$. *End of example.*

In the following, we define useful concepts related to processes. We introduce the following notation for the probability measure over outcomes derived from a process (G, θ):

$$\rho_{G,\theta} = \bigoplus_{\mu} \theta[\mu] \bigoplus_{a} \mu[a] o[a] \tag{6}$$

$$\rho_{G,\theta}|\mu_i = \bigoplus_{\mu_{-i}} \theta_{-i}[\mu_{-i}] \bigoplus_{a_i \in \mathcal{A}_i, a_{-i} \in \mathcal{A}_{-i}} \mu_i[a_i]\mu_{-i}[a_{-i}] o[a_i, a_{-i}] \tag{7}$$

For an outcome x, $\rho_{G,\theta}[x]$ thus denotes the probability of how likely it is that outcome x occurs in a game form G with information θ about the strategies played by players. The conditional probability given a particular strategy is defined analogously.

We now extend our notion of strategic equivalence to take into account the information θ of how players choose their strategies. For notational convenience, we again[11] use the symbol $\approx$.

Definition 15 (Outcome Equivalent Strategies)**.** Two strategies $\mu_i \approx_{G,\theta} \mu_i'$ of player i are outcome equivalent in (G, θ) if $\rho_{G,\theta}|\mu_i[o_i] = \rho_{G,\theta}|\mu_i'[o_i]$ for all $o_i \in \mathcal{O}_i$.

Thus, two strategies are outcome equivalent for player i, if their conditional probability over i's individual outcomes is identical.

Example. In the Montgomery Bus Game, player 1's mixed strategies $\mu_1 = \frac{1}{2}\mathbb{1}_{a_1^1} \oplus \frac{1}{2}\mathbb{1}_{a_1^3}$ and $\mu_1' = \frac{1}{2}\mathbb{1}_{a_1^2} \oplus \frac{1}{2}\mathbb{1}_{a_1^4}$ are outcome equivalent irrespective of player 2's behavior. Moreover, $\mathbb{1}_{a_1^1} \approx_{G,\theta} \mathbb{1}_{a_1^2}$ if player 2 plays a_2^3 with zero probability in (G, θ). Note that if $\mathcal{O}_2$ partitions the outcomes of the game into the trivial partition $\mathcal{O}_2 = \{\{u, v, w, x, y, z\}\}$, then all strategies of player 2 are outcome equivalent for player 2. *End of example.*

Definition 16 (Outcome Equivalent Processes)**.** Two processes $(G, \theta), (G', \theta')$ are outcome equivalent for player i, $(G, \theta) \approx_i (G', \theta')$, if there exists a bijection $b_i : (\Delta\mathcal{A}_i)/{\approx_{G,\theta}} \to (\Delta\mathcal{A}_i')/{\approx_{G',\theta'}}$, such that for all $\mathcal{M}_i \in (\Delta\mathcal{A}_i)/{\approx}$: $\theta_i[\mathcal{M}_i] = \theta_i'[b_i[\mathcal{M}_i]]$, and for all $\mu_i \in \mathcal{M}_i, \mu_i' \in b_i[\mathcal{M}_i], x_i \in \mathcal{O}_i$:

$$\rho_{G,\theta}|\mu_i[x_i] = \rho_{G',\theta'}|\mu_i'[x_i]. \tag{8}$$

Two processes $(G, \theta), (G', \theta')$ are outcome equivalent, $(G, \theta) \approx (G', \theta')$, if they are outcome equivalent for all players.

In other words, processes are outcome equivalent if the (equivalence classes of) strategies with the same conditional probability over individual outcomes can be matched such that their total probability according to θ, θ' are identical.

[11]It will always be clear from context what type of equivalence is meant.

We now define the corresponding notion of a subgame for a process. If a game has a subgame, then conditioning the probabilistic beliefs of the policy maker over strategies to that subgame results in a subprocess.

Definition 17 (Subprocess)**.** For a process (G, θ), let $\mathcal{B} = \prod_{i \in \mathcal{N}} \mathcal{B}_i$ be a normal form subgame of G. Then $(G, \theta)|\mathcal{B} = ((\mathcal{N}, \mathcal{B}, o|_{\mathcal{B}}), \theta')$ defined by

$$\forall i \in \mathcal{N}, \mu_i \in \Delta\mathcal{A}_i : \quad \theta'_i[\mu_i|\mathcal{B}_i] = \frac{\theta_i[\mu_i]\mu_i[\mathcal{B}]}{\sum_{\mu'_i} \theta_i[\mu'_i]\mu'_i[\mathcal{B}]} \tag{9}$$

is a subprocess of (G, θ) on $\mathcal{B}$.

Example. Suppose player 1 plays either a mixed strategy involving actions a_1^5 and a_1^1 or plays the pure strategy $\mathbb{1}_{a_1^5}$ according to the information of the policy maker. If the policy maker wants to separately analyze the subgame $\{a_1^1, \ldots, a_1^4\} \times \{a_2^1, a_2^2\}$, the policy maker has to take into account that the probabilities of the two strategies change; the second strategy never reaches the subgame and therefore the first strategy is played in the subprocess with certainty. Moreover, the mixed strategies must be conditioned on the subgame; in the first strategy, a_1^1 is played with certainty on the subgame. *End of example.*

Using the above definition of Outcome Equivalence, we can define what it means that two processes $(G, \theta), (G, \theta')$ agree outside a subgame $\mathcal{B} \subseteq \mathcal{A}$. Suppose for some $b \in \mathcal{B}$,

$$o''[a] = \begin{cases} o[a] & a \in \mathcal{A} \backslash \mathcal{B} \\ o[b] & a \in \mathcal{B}. \end{cases} \tag{10}$$

The two processes (G, θ), (G, θ') agree outside the subgame $\mathcal{B}$ if $((\mathcal{N}, \mathcal{A}, o''), \theta) \approx ((\mathcal{N}, \mathcal{A}, o''), \theta')$. Therefore two processes agree outside a subgame $\mathcal{B}$ if making all actions on this subgame equivalent yields two outcome equivalent processes.

In some cases, we may want to capture the uncertainty the policy maker faces about what process will arise from the policy. We therefore define mixtures of processes as follows.

Definition 18 (Process Mixture)**.** The mixture of two processes, (G, θ) and (G', θ'), is defined as $\alpha(G, \theta) \otimes (1 - \alpha)(G', \theta') = ((\mathcal{N}, \mathcal{A}'', o''), \theta'')$

with

$$\begin{aligned}\mathcal{A}'' =& \mathcal{A} \times \mathcal{A}' \\ \mathcal{A}_i'' =& \mathcal{A}_i \times \mathcal{A}_i' \\ o''[a, a'] =& \alpha o[a] \oplus (1-\alpha) o'[a'] \\ \theta''[\mu \otimes \mu'] =& \theta[\mu]\theta'[\mu']. \end{aligned} \tag{11}$$

The mixture weight α represents how likely the policy maker believes it is that the process (G, θ) will be played. We can alternatively interpret this as nature determining which process will be played after the players have determined their strategies in each process independently.

Example. The policy maker may be informed by data that strategies are race-dependent. According to the data, if player 1 is black, she plays the pure strategy $\mathbb{1}_{a_1^1}$ with certainty, if she is white, $\mathbb{1}_{a_1^3}$ instead. Player 2 plays $\mathbb{1}_{a_2^1}$ with certainty. Let the corresponding processes be (G, θ) and (G, θ'). Since behind the veil of ignorance, race is determined by nature's lottery according to some proportion α, the process $\alpha(G, \theta) \otimes (1-\alpha)(G, \theta')$ represents the policy maker's beliefs of the overall process after receiving the information that strategies are race dependent. *End of example.*

In the axiomatization, we will analyze changes in processes that involve informing the policy maker that a strategic choice was actually determined by nature. In some initial process, the policy maker is informed that some player makes a choice between strategies. Now imagine informing the policy maker with the exact same information, except that according to the new information, the player did not choose among two strategies but instead a random process (nature) made this choice. We call the change from the initial to the latter process a choice removal.

Definition 19 (Choice Removal)**.** Suppose $\mathcal{M}_i \subseteq supp[\theta_i]$ is a set of strategies of player i. Then $D_i^{\mathcal{M}_i}(G, \theta)$ randomizes the choice among these strategies.

$$D_i^{\mathcal{M}_i}(G, \theta) = \bigoplus_{\mu \in \mathcal{M}_i} \frac{\theta[\mu]}{\theta[\mathcal{M}_i]} \left(G, \theta_{-i} \otimes \left(\theta[\mathcal{M}_i] \mathbb{1}_\mu \oplus (1 - \theta[\mathcal{M}_i]) \theta | \mathcal{M}_i^C \right) \right). \tag{12}$$

If all strategies of a player are randomized, we denote $D_j^{\mathcal{A}_j} = D_j$. The choice removal for all players in set $\mathcal{N}' \subseteq \mathcal{N}$ is denoted by $D_{\mathcal{N}'}$ and $D_{-\mathcal{N}'} = D_{\mathcal{N} \setminus \mathcal{N}'}$.

We interpret the above notation as follows. The process inside the brackets is the mixed belief of the policy maker whether a particular strategy $\mu_i \in \mathcal{M}_i$ is being played or a strategy outside $\mathcal{M}_i$ is being chosen. The latter choice among strategies is made with the same probabilities with which the strategies outside $\mathcal{M}_i$ were chosen in (G, θ). Via nature's randomization over processes, nature determines which of these strategies $\mu_i \in \mathcal{M}_i$ is chosen.

Example. We continue the example of a process mixture. Suppose initially, the policy maker has no data on race being the determining factor in the strategic choice of player 1. Instead, the policy maker falsely believes that with probability α, player 1 chooses $\mathbb{1}_{a_1^1}$ and with probability $1 - \alpha$, player 1 chooses $\mathbb{1}_{a_1^3}$. In this case, the policy maker attributes the choice therefore to player 1. Let this process be denoted by (G, θ''). Then $D_i(G, \theta'') = \alpha(G, \theta) \otimes (1 - \alpha)(G, \theta')$. In other words, the difference between the process in which race determines the choice of the player and the process in which the choice is of player 1's own volition is the choice removal operation. *End of example.*

Some game forms are effectively lotteries from the perspective of some players. Such players have no meaningful strategic choice and therefore removing their strategic choice leaves an outcome equivalent process. We use this idea to capture whether a player is influential or not.

Definition 20 (Influential Players)**.** A player i is *influential* in process $(G, \theta) \in \mathcal{P}$ if $D_i(G, \theta) \not\approx (G, \theta)$. The set of all *influential* players is $\{i \in \mathcal{N} : D_i(G, \theta) \not\approx (G, \theta)\}$.

We conclude this section with a summary of introduced concepts. We defined processes as combinations of game forms with probabilistic information of the policy maker about the strategies players choose. Next, we introduced outcome equivalence as a way to determine similarity of player's strategies across game forms. Further, we introduced subprocesses as the corresponding concept to subgames. Finally, the choice removal removes agency from a player and hands it to nature. Choice removal allows us to express whether a player is influential or not.

4 Axiomatization

We phrase the problem of finding a measure of freedom as a problem of finding a representation of the policy maker's preference relation $\succsim$

over processes $\mathcal{P}$. Behind the veil of ignorance, the policy maker must decide which process to implement for the players and forms preferences according to certain desirable criteria described below. Under these criteria, we then obtain a representation defined as:

Definition 21 (Representation). A function $U : \mathcal{S} \to \mathcal{R}$ represents a binary relation $\succsim$ if for all $a, b \in \mathcal{S}$,

$$a \succsim b \Leftrightarrow U(a) \geq U(b). \tag{13}$$

U is called a representation of $\succsim$.

To ensure that the relation is nontrivial, we employ the following definition of essentiality:

Definition 22. A pair of social outcomes x, y is essential for player i if $\nexists x_i \in \mathcal{O}_i : x, y \in x_i$ and it is not the case that all processes on the set $\{(G, \theta) \in \mathcal{P} : G \in \mathcal{G}[\{x, y\}]\}$ are indifferent.

With the first axiom, we assume the policy maker has a complete and transitive preference relation on processes.

Axiom 1 (Rationality). $\succsim$ is a weak order on $\mathcal{P}$, i.e.,

— $a, b \in \mathcal{S}$ implies $a \succsim b$ or $b \succsim a$ or both.

— $a \succsim b$, $b \succsim c$ imply $a \succsim c$.

Transitivity is a natural requirement from a rationality perspective. However, completeness relies on the policy maker having to rank all possible processes. This is more restrictive as the policy maker may be unwilling to rank two processes on different decision domains. The policy maker may also find some game form G and some information θ incompatible with each other and may therefore be unable to compare (G, θ) to other processes.

Processes that are similar to each other should also be similarly ranked by the policy maker. We therefore adapt two conditions that ensure this. The following Continuity axiom ensures that convergence of information about behavior ensures convergence in preference of the policy maker.

Axiom 2 (Continuity). For all $p \in \mathcal{P}$ and all game forms G the lower and upper sets of $\succsim$, $\{\theta \in \Delta\Delta\mathcal{A} : p \succsim (G, \theta)\}$ and $\{\theta \in \Delta\Delta\mathcal{A} : (G, \theta) \succsim p\}$ are closed.

Continuity requires that for a fixed game form, similar information over players' strategies yields a similar ranking in the preference. It does not require that similar game forms are similarly ranked. For this, we assume the following Outcome Equivalence axiom.

Axiom 3 (Outcome Equivalence)**.** $(G, \theta) \approx (G', \theta') \Rightarrow (G, \theta) \sim (G', \theta')$.

Outcome equivalence makes different game forms comparable. Game forms only matter to the extent that they generate strategic choices with different conditional probabilities over individual outcomes.[12]

Example. In the Montgomery bus game, the policy maker may have some observational data about players' behavior. The policy maker however does not observe the action in which player 1 buys her own bus and drives by herself. Neither does the policy maker observe that player 1 constructs her own vehicle, goes by airplane etc.. However, most likely the policy maker cannot exclude with certainty that these options were not part of the game form when the choice was made. Outcome Equivalence handles this issue by imposing on preferences that changing the game form to allow for such actions does not change the preferences of the policy maker unless these actions are chosen with positive probability. This is the central advantage of ranking combinations of game forms with information about behavior instead of game forms. Representing policies by game forms would probably also require the choice of the game form to contain implicitly the policy maker's beliefs about which actions are played. Alternatively, one would need to rely on definitions such as what the *legal* or *possible* actions are. In the Montgomery bus game, including only *legal* actions would remove the actually played actions a_2^2 and a_2^3. Allowing for all *possible* actions would add the action of player 1 to construct her own vehicle. Neither of these options seems attractive for the purposes of policy evaluation. *End of example.*

We now impose independence conditions on three levels. First, on lotteries over outcomes, second on the probabilistic information over strategies, and third on subprocesses. The axiom on the independence of lotteries over outcomes is the standard von Neumann-Morgenstern axiom adapted to our setting. Processes in which no player is influential are effectively lotteries and thus we apply the independence axiom with respect to such processes.

[12]Note that Outcome Equivalence creates large equivalence classes of processes. It may be interesting to consider changes to this axiom to measure other aspects such as power or information transmission in games.

Axiom 4 (Lottery Independence)**.** Suppose no player is influential in (G, θ), (G, θ'), (G, θ''), then,

$$
\begin{aligned}
& (G, \theta) \succsim (G, \theta') && (14)\\
\Leftrightarrow \quad & \alpha(G, \theta) \oplus (1 - \alpha)(G, \theta'') \succsim \alpha(G, \theta') \oplus (1 - \alpha)(G, \theta'') && (15)
\end{aligned}
$$

In other words, if nature fully controls the outcomes of the players, then the standard independence axiom holds. In combination with Rationality and Continuity, this axiom requires the decision maker to have expected utility preferences over pure lotteries, i.e., processes in which nature determines the outcomes and players have no meaningful choice. Lottery Independence excludes certain value judgments about institutions. Most importantly, it does not allow for source-dependent attitudes towards risk (Chew & Sagi, 2008) of the policy maker.

Example. Consider nature's lottery over race and gender in the Montgomery Bus Game. A policy maker who finds outcome-dependence on gender undesirable but outcome-dependence on race acceptable violates Lottery Independence. Suppose (G, θ) is a process in which player 1 is a black female and (G, θ') is the process in which player 1 is a black male. In each process, the policy maker believes that player 1 plays a_1^1 with certainty and player 2 plays a_2^1. The policy maker is indifferent between the two processes. Now suppose (G, θ'') is the process of a white male who chooses a_1^3. Then, the LHS mixture of (15) is a gender-and-race-dependent lottery while the RHS is a race-dependent lottery. Since gender-dependent lotteries are intrinsically undesirable to the policy maker, the indifference no longer holds – the policy maker prefers the RHS mixture to the LHS, violating Lottery Independence. *End of example.*

Strategy Independence ensures that the value of choice across strategies is independent across the different strategies.

Axiom 5 (Strategy Independence)**.** Suppose $\theta_i[\mu_i] = \theta_i'[\mu_i]$ and $\rho_{G,\theta}|\mu_i = \rho_{G,\theta'}|\mu_i$ for all $\mu_i \in \mathcal{M}_i \subseteq \Delta A_i$, then

$$
(G, \theta) \succsim (G, \theta') \quad \Leftrightarrow \quad D_i^{\mathcal{M}_i}(G, \theta) \succsim D_i^{\mathcal{M}_i}(G, \theta') \qquad (16)
$$

In other words, the value of choice between two strategies does not depend on the other choices being made. The choice removal $D_i^{\mathcal{M}_i}$ has the effect of taking the choice between some strategies $\mathcal{M}_i$ out of the control of player i. From the perspective of all other players, the game remains unchanged; the probability of each outcome given any of their strategies is the same as before choice removal.

Example. Suppose the policy maker believes that player 1 chooses with equal probability to play the pure strategies $\mathbb{1}_{a_1^1}$, $\mathbb{1}_{a_1^3}$, or $\mathbb{1}_{a_1^5}$. In another process, the policy maker believes that player 1 chooses with equal probability $\mathbb{1}_{a_1^1}$, $\mathbb{1}_{a_1^3}$, or $\mathbb{1}_{a_1^4}$. Suppose that the policy maker learns that the choice between the pure strategies of $\mathbb{1}_{a_1^1}$, or $\mathbb{1}_{a_1^3}$ is determined by race. Then the ranking between the two processes remains unchanged if the behavior of the other player is identical in both processes. The latter requirement is crucial; in case player 2 plays the pure strategy a_2^3, the choice between $\mathbb{1}_{a_1^1}$ and $\mathbb{1}_{a_1^3}$ is meaningless. In case player 2 plays a_2^1, then the choice of player 1 is effectively between outcomes u_1 and v_1. Strategy Independence therefore captures that ceteris paribus the value of making a strategic choice instead of nature determining the choice does not depend on other strategic choices. *End of example.*

Next, Subprocess Monotonicity ensures that the value of choice across subprocesses is independent if the outcomes of the subgame are independent of the remainder of the game.

Axiom 6 (Subprocess Monotonicity)**.** Let (G, θ) and (G, θ') be equivalent outside the non-null subgame $\mathcal{B}$. Suppose the set of influential players in both processes is $\mathcal{N}'$ and for all $i \in \mathcal{N}'$ we have that $supp_i[o[\mathcal{B}]] \cap supp_i[o[\mathcal{A} \setminus \mathcal{B}]] = \emptyset$. Then,

$$(G, \theta) \succsim (G, \theta') \Leftrightarrow (G, \theta)|\mathcal{B} \succsim (G, \theta')|\mathcal{B}. \tag{17}$$

Put simply, the relation $\succsim_i$ is monotone in subprocesses that have a disjoint support from the remainder of the game form: we can improve a process by improving any subprocess unless some of the outcomes of the subprocess are identical to the remainder of the process. The central idea behind this axiom is the following: if a set of players is influential, they can make choices to favor their interests. If they can better favor their interests in a subgame, then this is preferable from the perspective of the policy maker. However, this is only the case if this improvement does not come at the cost of influence across the entire game.

Example. This example illustrates the need for requiring disjoint outcomes. Suppose only player 1 is influential and is choosing between sitting and standing on the bus via the pure strategies $\mathbb{1}_{a_1^1}$ and $\mathbb{1}_{a_1^3}$. Let's suppose the policy maker is indifferent between disallowing standing on the bus or disallowing sitting on the bus (essentially, limiting the player to either of the two actions). Since outcomes are subgames, any game involving a choice between standing and sitting contains the subgame in which the player stands or sits with certainty. Under Subprocess Monotonicity *without* requiring disjoint outcomes, replacing the sitting subgame by the subgame in which the player stands would leave the policy maker indifferent. But then the agent is left in the overall game with a trivial decision between standing and standing. This means that a meaningful choice (between sitting and standing) and a meaningless choice (between standing and standing) are equally good according to the preferences of the policy maker.

The problem can be resolved by requiring disjoint outcomes in Subprocess Monotonicity. Replacing the subgame in which the player sits by a subgame in which the player stands does not leave the policy maker indifferent in case somewhere in the remainder of the process the player stands. The monotonicty axiom in this case does not bind since the outcomes of the two subprocesses are not disjoint. *End of example.*

In the appendix, we prove the following theorem.

Theorem 1. *Suppose for every player i there are at least four essential pairs of outcomes. The relation $\succsim$ on the process space $\mathcal{P}$ fulfills Axioms 1-6 if and only if there exists a continuous, real valued representation $U : \mathcal{P} \to \mathbb{R}$ such that*

$$U[G,\theta] = \sum_{i \in \mathcal{N}} U_i[G,\theta] \tag{18}$$

$$U_i[G,\theta] = \sum_{\mu_i} \theta_i[\mu_i] \sum_{x_i \in \mathcal{O}_i} \rho_{G,\theta}|\mu_i[x_i] \left(u_i[x_i] + d_i \ln \left[\frac{\rho_{G,\theta}|\mu_i[x_i]}{\rho_{G,\theta}[x_i]} \right] \right) \tag{19}$$

For each player, the measure consists of an expected valuation of the outcomes and the player's control over own individual outcomes. We call the expected valuation the instrumental value of the process and the control measure the freedom value of the process. We denote the freedom measure of player i by $F_i[G,\theta] = U_i[G,\theta] - U_i[D_i(G,\theta)]$. Freedom is measured by the mutual information between strategies and outcomes. Mutual information is a measure of correlation that imposes no structure on the relation between variables. In comparison, the correlation coefficient assumes a linear relation. Spearman's rank order correlation assumes that each of the variables can be ordered. Since the policy maker is given no information about the intention behind strategies, mutual information is the adequate correlation measure for judging the degree to which players use strategies to control outcomes.

Example. We return to the Montgomery bus game to exemplify the data necessary to apply the measure. As stated before, the advantage of our framework is that it does not rely on unobservable information about utilities. Instead, we require observational or experimental data about the strategies of the players that can help inform the policy maker.[13] For simplicity, we focus on the subprocess induced by the subgame $\{a_1^1, \ldots, a_1^4\} \times \{a_2^1, a_2^2\}$. This is meaningful thanks to the Subprocess Monotonicity axiom; if we are interested in the freedom of choice of the overall process, we would simply need to calculate the freedom of choice from the remaining process with an arbitrary outcome u substituted for the subgame in which player 1 boards the bus. To the freedom obtained for this process we then add the sum of the freedom from the subprocess times the probability of reaching this subprocess. More succinctly put, the freedom of an overall process consists of the expected freedom of the disjoint subprocesses plus the freedom derived from determining which of these subprocesses is played. While this is not obvious from the functional form of the freedom measure, it is guaranteed via Subprocess Monotonicity. This decomposability property is important for applications since it allows us to focus on localized data specific to an interaction.

The reduced form of the subgame contains the action profiles $\{\{a_1^1, a_1^2\}, \{a_1^3, a_1^4\}\} \times \{a_2^1, a_2^2\}$. Under the institutional setting of Montgomery in the early 1950s, we can inform a policy maker using the following information. To know the weight attached to nature's lottery over race, the policy maker needs to know the ridership composition, or more precisely the frequency with which individuals of different background play the Montgomery bus game. Next, we need to know the strategies chosen by the players, conditional on race. An extremely small fraction of courageous black women took the action $\{a_1^3, a_1^4\}$.[14]. Since in their cases the driver took action a_2^2, they were arrested. The overwhelming majority of black passengers endured the discriminatory treatment by taking action $\{a_1^1, a_1^2\}$. To account for cases in which player 1 refused to yield their seat and managed to keep the seat, we would also need to obtain information whether any drivers took action a_2^1 when interacting with a black passenger. Lacking evidence of such cases, we assume that

[13] The central difficulty of applying the measure to observational data is the possible existence of mixed strategies. In the Montgomery bus game there is no good reason to assume that players play mixed strategies but in other games this might be different. Then observational data of the actions taken is not sufficient for the estimation of freedom of choice since the choice is between mixed strategies, not actions. Instead, the mixed strategies would need to be identified via experimental treatments.

[14] Claudette Colvin, Aurelia Browder, Susie McDonald, Mary Louise Smith, Jeanetta Reese, and Rosa Parks.

this did not happen. Similarly, there are no known accounts of white passengers being arrested for refusing to give up their seat. To estimate their freedom of choice of voluntarily yielding their seat, we need to know the fraction of white passengers that take action a_1^1. If α is the fraction of black passengers, ϵ the fraction of black passengers refusing to yield their seat and being arrested, and γ the fraction of white passengers yielding their seat voluntarily, then the non-instrumental component of freedom of choice yields after rearranging terms:

$$
\begin{aligned}
& (U_1[G,\theta] - U_1[D_1(G,\theta)])/d_i \\
= \quad & \epsilon \cdot \alpha \left(\ln \frac{1}{\epsilon} \right) \\
& + (1-\gamma)(1-\alpha) \left(\ln \frac{1}{1-\gamma} \right) \\
& + (\alpha(1-\epsilon) + (1-\alpha)\gamma) \left(\ln \frac{1}{\alpha(1-\epsilon) + (1-\alpha)\gamma} \right) \\
& + \epsilon\gamma(1-\alpha) \ln (1-\alpha) + (1-\epsilon)(1-\gamma)\alpha \ln \alpha
\end{aligned}
\tag{20}
$$

The measure is monotonically increasing in ϵ for the given scenario in which ϵ is small. A central distinction from impartial observer theorems is that the freedom of player 1 is not the weighted sum of the freedom of a white player 1 and a black player 1. The fact that the outcomes are partially determined by nature's lottery over race is intrinsically undesirable which prevents this separability. In the following, we interpret the terms in (20). The first three rows of (20) are the entropy of the outcomes reached. In case player 1 would have perfect control over which of the outcomes u, v, w arises, this would be the freedom of choice of the player.[15] However, the outcome partially depends on nature's lottery. The last row of (20) corrects for this with two negative terms. The first term corrects for the probability $1-\alpha$ with which player 1 reaches outcome u when "choosing" outcome w. The second term corrects for the probability with which player 1 reaches w when "choosing" u. Both terms are negative since α and $1-\alpha$ are smaller than one. In each case, the correction term arises from the fact that the conditional probability of the outcome given the strategy is not equal to one.

The example also shows that the freedom measure is completely neutral towards the characteristic of the outcomes. All judgments regarding whether for example the outcome v is in the eyes of the policy maker more desirable than the outcome w can only be contained in the

[15] Indeed, this would be the Suppes (1996) measure.

instrumental value of the process. *End of example.*

It is important to note that the instrumental value u_i does not depend on the strategies by which this instrumental value is reached. This means that –to the extent that utilities are only meaningful in their impact on behavior– all potential utility information is accounted for outside of u_i! In other words, if behavior is a sufficient statistic for utility (as is commonly assumed in economics), then the axiomatization guarantees that u_i contains no utility information. This has far reaching consequences for a utilitarian policy maker. Either the policy maker accepts that in the absence of cardinal utility information the mutual information criterion is the best we can do to *approximate* utilitarianism or the policy maker must argue why any of the stated axioms must be violated in the estimation of the expected utility of every player.

A commonly employed assumption in the freedom of choice literature removes the dependence of freedom of choice on the policy maker's norms by assuming that all singletons are indifferent. We can translate this condition into our setting as saying that all outcomes are instrumentally equally valuable to the policy maker:

Remark 1. Suppose we assume the assumption of the indifference of no-choice axiom (Pattanaik & Xu, 1990), i.e., for all social outcomes x, y, the trivial games yielding the outcomes with certainty are indifferent, $((\mathcal{N}, \{a\}, a \mapsto x), \mathbb{1}_{\mathbb{1}_a}) \sim ((\mathcal{N}, \{a\}, a \mapsto y), \mathbb{1}_{\mathbb{1}_a})$. Then the instrumental value is a constant and freedom only depends on the mutual information between strategies and individual outcomes of every player.

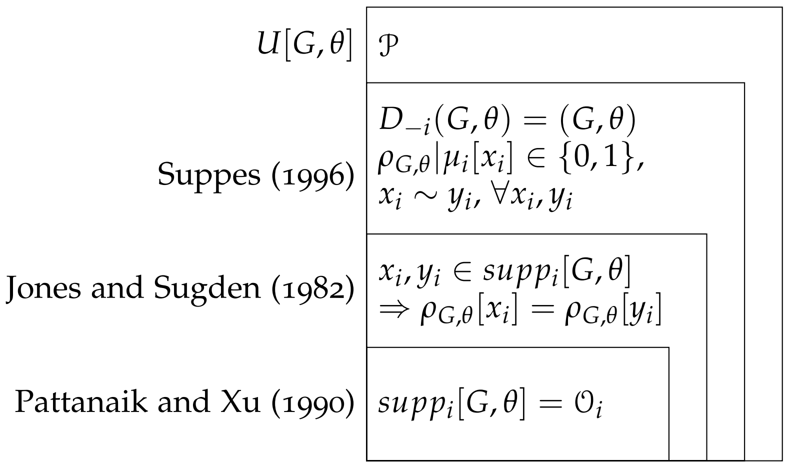

Figure 1: Relation of Freedom Measures

The axiomatized measure generalizes the entropy measure (Suppes,

1996) of freedom. This follows since the entropy is a limiting case of mutual information.

Remark 2. Suppose indifference of no-choice holds and only player i is influential. Moreover, let $supp(\theta_i)$ contain only pure strategies and for all a, $supp[o[a]] \subseteq x_i$ for some $x_i \in \mathcal{O}_i$. Then, $U[D_{-i}(G,\theta)] = F_S((x \mapsto x_i)\sharp\rho_{G,\theta})$.

This establishes the relation of the measure to the Suppes measure. In case an individual has perfect control over outcomes and the policy maker is indifferent between all outcomes, then the measure is equal to the Suppes measure of the distribution of the probability distribution over outcomes. The result follows from the fact that $supp[o[a]] \subseteq x_i$ ensures $\rho|\mathbb{1}_a[x_i] = 1$. From this result, the comparisons to the cardinality measure and the reasonable preference measure directly follow. For convergence to the reasonable preference measure, we would need to impose that the information of the policy maker θ is information about what reasonable players would choose. Moreover, all outcomes in the support must be equally likely. For convergence to the cardinality measure, we would need to additionally impose that all outcomes are in the support. Figure 1 displays the relation between different measures.

5 Freedom in a Production Economy

The utility of a definition naturally rests in how useful it is in applications. To further illustrate the defined freedom measure, we analyze freedom in a production economy, a similar problem to the one put forward in Pattanaik (1994). According to Pattanaik (1994), the problem of measuring freedom in an exchange economy is that prices and therefore also opportunity sets change both with one's own preferences and preferences of the other agents. Since most measures of freedom are based on opportunity sets, they fail to give a satisfying answer to the problem, as Pattanaik (1994) concludes. The following subsections develop a production economy with heterogeneous consumers differing in both tastes and productivity. From the perspective of the policy maker, each individual's tastes and production possibilities are uncertain. More precisely, the policy maker does not know the individual's expected utility functions. Thus, no cardinal value can be attached to utilities. Instead, utilities are only used to rationalize behavior, i.e., to generate predictions about consumer behavior. The policy maker forms preferences over different levels of tax progression. Ex ante, it is unclear whether high or low redistribution is optimal for freedom. A politician in favor of higher redistribution might argue that income differences limit freedom and redistribution gives everybody equal opportunities. A politician in favor of less redistribution might argue that redistribution intervenes in the personal decisions of individuals to consume more or less by distorting working incentives. Using the freedom measure, we can disentangle these qualitative intuitions.

5.1 Production Economy

According to the information of the policy maker, the economy is structured as follows. A continuum of individuals indexed by i is uniformly distributed on the unit interval $[0,1]$. Individuals have preferences over consumption x_i and labor effort y_i. To obtain closed-form solutions, parametric assumptions on the preferences over consumption $x_i \geq 0$ and labor effort $y_i \geq 0$ are made:[16]

$$u_i(x_i, y_i) = \alpha_i \frac{(\delta x_i)^{1-\eta}}{1-\eta} - \frac{y_i^{\zeta+1}}{\zeta+1}. \tag{21}$$

where $\eta \in (0,1)$ and $\zeta \in (0,\infty)$. The individual specific preference parameter α_i is distributed log-normally and independent across individuals, i.e., $\alpha_i \sim \ln \mathcal{N}(\mu_\alpha, \sigma_\alpha^2)$.[17] Quality δ is measured by the quantity of unit-quality goods that leaves the consumer indifferent to one unit of quality δ. δ is distributed log-normally, $\delta \sim \ln \mathcal{N}(\mu_\delta, \sigma_\delta^2)$.

Let there be a competitive firm with production function $Q = \int_i y_i \beta \gamma_i di$. The economy-wide efficiency parameter β is distributed log-normally with $\beta \sim \ln \mathcal{N}(\mu_\beta, \sigma_\beta^2)$. Similarly, γ_i is distributed log-normally with $\gamma_i \sim \ln \mathcal{N}(\mu_\gamma, \sigma_\gamma^2)$. The firm's profit is given by $pQ - \int w_i y_i di$. In equilibrium, the firm earns zero profits and wages and prices are $w_i = p\beta\gamma_i$.

We introduce a government that taxes income via a progressive tax. The net income of each individual i equals expenditure, $\frac{f(r,\beta,\delta)(y_i w_i)^{1-r}}{1-r} = x_i p$. The parameter r determines the progressivity of the tax and $f(r,\beta,\delta)$ is chosen such that the government balances its budget. The government consumption is assumed to equal a fixed total share $\bar{g}$ of the output of the firm, $g = \bar{g}Q$ and therefore $(1-\bar{g})Q = \int x_i di$. In the appendix

[16] For ease of calculations, we assume that $y_i \in (0,\infty)$.

[17] Technically, the assumption here is that there exists a measure on a continuum of random variables such that the weak law of large numbers holds and each variable is i.i.d. lognormal. For a general existence proof of such measures, see Theorem 2 of Judd (1985).

it is shown that consumption demand and labor supply in the above described economy are (up to a proportionality factor):

$$x_i^* \propto \left((\alpha_i \delta^{1-\eta})^{1-r} \left(\beta^{1+r\frac{1-\eta}{\zeta+\eta}} \gamma_i^{1-r} \right)^{\zeta+1} \right)^{\frac{1}{\zeta+\eta(1-r)+r}} \tag{22}$$

$$y_i^* \propto \left(\alpha_i \delta^{1-\eta} \left(\beta^{1+r\frac{1-\eta}{\zeta+\eta}} \gamma_i^{1-r} \right)^{1-\eta} \right)^{\frac{1}{\zeta+\eta(1-r)+r}} \tag{23}$$

5.2 The Production Economy as a Process

Although a production economy is commonly not perceived as a game, we can still model it as a process.[18] We define the set of players $\mathcal{N}$ as the unit interval of individuals.

Let the set of actions of a player be the set of demand functions, $\mathcal{A}_i = \{(p, w) \mapsto (x_i, y_i) : x_i[p, w]p = y_i[p, w]w\}$, i.e., the available actions of each individual are all feasible consumption demand and labor supply functions given the budget constraint. We assume that when the individual chooses the demand function, he is informed about the quality of the good δ, his productivity γ_i and the aggregate shock β. Thus, a strategy is a fully contingent plan in which each individual i chooses a demand function for each possible value of γ_i and β. According to the information of the policy maker, the support of θ_i contains only the strategies with supply and demand functions consistent with maximization of (21). Since for every value of α_i there exists a unique strategy, the set of distinct strategies in the support of θ_i has a real valued representation in the form of α_i. θ_i is such that each α_i follows a lognormal distribution. A strategy profile is therefore represented by a mapping $\alpha \in \mathbb{R}_+^{\mathcal{N}}$. This identification of strategies with a preference parameter is the main conceptual step in translating the production economy into a process.

What is left to do is to ensure that the conditional distribution of allocations given strategies is consistent with that in the production economy. To this end, $o : \mathcal{A} \to \mathcal{O}$ yields a distribution over social outcomes as follows. Each social outcome is an allocation $(y^* : \mathcal{N} \to \mathbb{R}_+, x^* : \mathcal{N} \to \mathbb{R}_+)$ that is randomly determined via a measure $o[\alpha]$ consistent with Equations (23) for all individuals with lognormal distributions of β, γ_i, and δ. This concludes our translation of the production economy into a process and we can now apply the measure to this process.

[18] For a survey of game theoretic analyses of markets, see Giraud (2003).

To measure freedom, some normative impositions must be made regarding the measure. For clarity of the analysis, we ignore the tradeoff between the instrumental value and the freedom measure by assuming all individual outcomes are equally valuable from the perspective of the policy maker. This means we assume the *indifference of no-choice situations* axiom (Pattanaik & Xu, 1990) which allows us to focus on the mutual information, the freedom measure. Another simplifying assumption is that we value every individual's freedom identically $d_i = d_j$ for all $i, j \in [0,1]$.

The most interesting normative decision of the policy maker is the choice of the individual outcome partition $\mathcal{O}_i$. The choice of $\mathcal{O}_i$ is ultimately a normative question and we will explore in detail how the choice of the individual outcome space affects the preferences of the policy maker over different institutional arrangements.

We calculate freedom for three definitions of normatively relevant outcomes: consumption, labor, or combinations of consumption and labor. We may call the different freedoms consumption freedom, labor freedom, and demand freedom. Consumption freedom refers to the degree to which the consumer's strategy influences the quantity consumed. Labor freedom refers to the degree to which the consumer's strategy influences the amount of labor. Demand freedom refers to the degree to which the strategy determines combinations of consumption and labor.

Since the strategies of the individuals and the measure over outcomes are all continuous random variables, we replace the sums in the freedom measure by integrals, yielding the density mutual information.

5.3 Consumption Freedom

In this subsection the policy maker imposes that for each individual i, any two social outcomes (allocations) (x, y) and (x', y') in $\mathcal{O}$ are equivalent from the perspective of player i, if and only if the consumption of i is identical, $x_i = x'_i$. Therefore, the set $\mathcal{O}_i$ is the partition of $\mathcal{O}$ into sets of outcomes within which the consumption level of i is identical. We call this consumption freedom because the policy maker ignores all other differences in outcomes.

Proposition 1. *Consumption Freedom is measured by:*

$$\frac{1}{2}\ln\left[1 + \frac{(1-r)^2\sigma_\alpha^2(\eta+\zeta)^2}{(\eta+\zeta-\eta r+r)^2((\zeta+1)^2\sigma_\beta^2+(1-\eta)^2\sigma_\delta^2)+(1-r)^2(\eta+\zeta)^2(\zeta+1)^2\sigma_\gamma^2}\right] \tag{24}$$

From this proposition it becomes apparent that consumer freedom is increasing in σ_α^2 and decreasing in σ_β^2, σ_γ^2, and σ_δ^2. This is intuitive, σ_α^2 increases the diversity of strategies played, increasing freedom. For a very small variance of α, the individual is almost completely determined in his preferences over consumption and labor effort. If the variance of α is large, the individual may also prefer very high consumption and high labor effort or low consumption and low effort. This effect could also be observed in the opportunity set based measure of Jones and Sugden (1982) in a deterministic setting: for a given opportunity set, the measure increases when adding additional reasonable preference relations with new optimal elements of the opportunity set.

σ_β^2, σ_γ^2, and σ_δ^2 decrease the control of the consumer by stochastically "disturbing" the budget constraint and the quality of the received good. This is intuitive since stochastic production possibilities limit the extent to which an individual's preferences control consumption. An individual is less free, if his consumption strongly dependens on fluctuating production conditions, including quality. Nonetheless, if the policy maker has the choice to implement any degree of tax progression, the policy maker will choose a flat tax, $r = 0$. This is because tax progression not only removes the effect of individual productivity disturbances γ_i but also decreases strategic diversity arising from different α_i; the strategies played by two different realizations of α_i become more similar with higher r. In the case of $\sigma_\beta^2 = 0$, these effects cancel exactly and any level of progression is optimal. However, for positive σ_β^2, tax progression is not fully effective at reducing productivity disturbances as it attenuates disturbances on the economy-wide level less effectively than at the individual level. The reduction in strategic diversity therefore dominates the reduction in productivity disturbances.

5.4 Labor Freedom

In this subsection we now assume that the policy maker only cares about the freedom to choose between allocations with different labor efforts. Therefore, (c, y) and (c', y') are treated as distinct outcomes for player i if and only if $y_i \neq y_i'$ and are otherwise elements of the same element of $\mathcal{O}_i$.

Proposition 2. *Labor Freedom is measured by:*

$$\frac{1}{2}\ln\left[1+\frac{\sigma_\alpha^2(\eta+\zeta)^2}{(1-\eta)^2\left(\left(\sigma_\beta^2+\sigma_\delta^2\right)(\eta+\zeta-\eta r+r)^2+(1-r)^2\sigma_\gamma^2(\eta+\zeta)^2\right)}\right] \tag{25}$$

For labor freedom, the comparative statics with respect to σ_α^2, σ_β^2, σ_γ^2, and σ_γ^2 are unchanged compared to consumption freedom; freedom increases in σ_α^2, but decreases in σ_β^2, and σ_γ^2. However, the strategic diversity has a direct effect on labor choices that is not mediated by $(1-r)$. It follows that labor freedom can be effectively increased using a progressive tax system. The policy maker faces a tradeoff between reducing inequality and leaving room for individuals to determine their labor outcomes that yield the following optimal solution:

Proposition 3. *The policy maker's optimal tax progressivity is given by:*

$$r^* = \min\left[1, \max\left[0, \frac{\sigma_\gamma^2(\eta+\zeta)^2-(\sigma_\beta^2+\sigma_\delta^2)(1-\eta)(\eta+\zeta)}{(\sigma_\beta^2+\sigma_\delta^2)(\eta-1)^2+\sigma_\gamma^2(\eta+\zeta)^2}\right]\right]. \tag{26}$$

On the interior, r^* is decreasing in σ_β^2, and increasing in σ_γ^2, ζ, and η.

Thus, if the policy maker expects disturbances in productivity mostly being on the individual level, the policy maker chooses a more progressive tax system. If the policy maker expects disturbances to be macroeconomic, the policy maker chooses a lower degree of tax progression.

5.5 Demand Freedom

If the policy maker imposes that two outcomes (c, y) and (c', y') are normatively equivalent from the perspective of individual i if and only if $x_i = x_i'$ and $y_i = y_i'$, we speak of a model of demand freedom. Since demand freedom uses the cartesian product of consumption and labor, demand freedom is the easiest to increase; a lack of freedom over consumption can be substituted for by a lack of freedom over labor and vice versa. It is harder to achieve independent control of the two variables.

The freedom over demand curves is exercised under perfect control except for the disturbance due to quality. This is evident from the demand equation

$$\alpha_i x_i^{1-\eta} \delta^{1-\eta} = \frac{y_i^{\zeta+1}}{1-r} \tag{27}$$

which depends on δ. Therefore, if the policy maker cares about freedom of choice over demand curves, we obtain that the policy maker will try to minimize the extent of the disturbance due to quality fluctuation; the freedom measure becomes:

Proposition 4. *Demand freedom is equal to:*

$$\frac{1}{2} \ln \left[1 + \frac{\sigma_\alpha^2}{(\eta-1)^2 \sigma_\delta^2} \right]. \tag{28}$$

We note that demand freedom is unaffected by the tax policy. Example policy instruments to achieve a higher freedom are consumer protection regulations and product standards. Such policies may reduce the stochasticity of quality σ_δ^2 and thereby increase freedom of choice. Naturally, such policies may also be instrumentally valued for other reasons. However, within the framework presented in this paper, we do not need to employ utilitarianism to motivate such policies.

Naturally, in the general context of an economy we would expect many other freedoms to play a role, for example the freedom to choose a specific occupation or to choose among different products. The model presented here is only a small starting point for a more general analysis of freedom of choice in markets.

6 Concluding Remarks

The policy evaluation criterion we presented is consistent with three principles that are commonly employed in economics. First, the criterion only depends on observable information; in classical welfare economics this information used is ordinal preference, in our game theoretic setting, the information used is the strategy of a player. Second, the criterion is non-paternalistic and makes individuals the best judges of what is desirable for themselves. This imposition is normative, in particular it does not depend on the rationality of players. Third, in game forms the criterion obeys independence across (disjoint) subgames. Without this independence, we would need to worry that improving freedom of choice in some context negatively impacts overall freedom of choice.

Taken together, these principles guarantee that the the criterion can readily be applied to many contexts, two of which have been exemplified in this paper.

References

Aczél, J. & Dhombres, J. (1989). *Functional Equations in Several Variables*. Cambridge, UK: Cambridge University Press.

Ahlert, M. (2008). Guarantees in Game Forms. In M. Braham & F. Steffen (Eds.), *Power, Freedom, and Voting* (pp. 325–341). Berlin, Heidelberg: Springer Berlin Heidelberg. doi:10.1007/978-3-540-73382-9_17

Ahlert, M. (2010, September). A new approach to procedural freedom in game forms. *European Journal of Political Economy*, *26*(3), 392–402. doi:10.1016/j.ejpoleco.2009.11.003

Arlegi, R. & Dimitrov, D. (2009). On Procedural Freedom of Choice. *Homo Oeconomicus*, *26*(2), 215–234. doi:10.2139/ssrn.556966

Barberà, S., Hammond, P. J., & Seidl, C. (Eds.). (2004). *Handbook of Utility Theory*. Boston, MA: Springer US.

Baujard, A. (2007). Conceptions of freedom and ranking opportunity sets: A typology. *Homo Oeconomicus*, *24*(2), 231–254.

Bavetta, S. (2004). Measuring freedom of choice: An alternative view of a recent literature. *Social Choice and Welfare*, *22*, 29–48.

Bavetta, S. & del Seta, M. (2001). Constraints and the measurement of freedom of choice. *Theory and Decision*, *50*, 213–238.

Bavetta, S. & Peragine, V. (2006). Measuring autonomy freedom. *Social Choice and Welfare*, *26*, 31–45.

Berlin, I. (1958). *Two concepts of liberty*. Oxford, UK: Oxford University Press.

Bervoets, S. (2007, July 25). Freedom of choice in a social context: Comparing game forms. *Social Choice and Welfare*, *29*(2), 295–315. doi:10.1007/s00355-006-0205-0

Bossert, W. (1998). Opportunity Sets and the Measurement of Information.

Braham, M. (2006, November). Measuring Specific Freedom. *Economics and Philosophy*, *22*(3), 317–333. doi:10.1017/S0266267106001003

Burns, S. (2012). *Daybreak of Freedom: The Montgomery Bus Boycott*. Univ of North Carolina Press.

Carter, I. (1992). The Measurement of Pure Negative Freedom. *Political Studies*, *40*, 38–50.

Carter, I. (1995). Interpersonal Comparisons of Freedom. *Economics and Philosophy*, *11*(1), 1–23. doi:10.1017/S0266267100003205

Carter, I. (2004). Choice, freedom, and freedom of choice. *Social Choice and Welfare*, *22*(1), 61–81. doi:10.1007/s00355-003-0277-z

Carter, I. (2013). Social Power and Negative Freedom. In M. J. Holler & H. Nurmi (Eds.), *Power, Voting, and Voting Power: 30 Years After* (pp. 27–62). Berlin, Heidelberg: Springer Berlin Heidelberg. doi:10.1007/978-3-642-35929-3_2

Chew, S. H. & Sagi, J. S. (2008, March). Small worlds: Modeling attitudes toward sources of uncertainty. *Journal of Economic Theory*, *139*(1), 1–24. doi:10.1016/j.jet.2007.07.004

Dowding, K. & van Hees, M. (2009). Freedom of Choice. In *Handbook of Rational and Social Choice*. Oxford, UK: Oxford University Press.

Ebanks, B. R., Kannappan, P., & Ng, C. T. (1987, December). Generalized fundamental equation of information of multiplicative type. *Aequationes Mathematicae*, *32*(1), 19–31. doi:10.1007/BF02311295

Elmes, S. & Reny, P. J. (1994). On the Strategic Equivalence of Extensive Form Games. *Journal of Economic Theory*, *62*, 1–23.

Erzurumluoglu, A. M., Liu, M., Jackson, V. E., Barnes, D. R., Datta, G., Melbourne, C. A., . . . Howson, J. M. M. (2019, January 7). Meta-analysis of up to 622,409 individuals identifies 40 novel smoking behaviour associated genetic loci. *Molecular Psychiatry*. doi:10.1038/s41380-018-0313-0

Gajdos, T. & Kandil, F. (2008, August). The ignorant observer. *Social Choice and Welfare*, *31*(2), 193–232. doi:10.1007/s00355-007-0274-8

Gilboa, I., Samet, D., & Schmeidler, D. (2004, August). Utilitarian Aggregation of Beliefs and Tastes. *Journal of Political Economy*, *112*(4), 932–938. doi:10.1086/421173

Giraud, G. (2003, July). Strategic market games: An introduction. *Journal of Mathematical Economics*, *39*(5-6), 355–375. doi:10.1016/S0304-4068(03)00049-1

Gorman, W. M. (1968, October 1). The Structure of Utility Functions. *The Review of Economic Studies*, *35*(4), 367–390. doi:10.2307/2296766. JSTOR: 2296766

Grant, S., Kajii, A., Polak, B., & Safra, Z. (2010). Generalized Utilitarianism and Harsanyi's Impartial Observer Theorem. *Econometrica*, *78*(6), 1939–1971. doi:10.3982/ECTA6712

Gustafsson, J. E. (2010, June). Freedom of choice and expected compromise. *Social Choice and Welfare*, *35*(1), 65–79. doi:10.1007/s00355-009-0430-4

Harsanyi, J. C. (1953a). Cardinal Utility in Welfare Economics and in the Theory of Risk-taking. *Journal of Political Economy*, *61*(5), 434–435. doi:10.1086/257416

Harsanyi, J. C. (1953b). Welfare Economics of Variable Tastes. *The Review of Economic Studies*, *21*(3), 204. doi:10.2307/2295773

Harsanyi, J. C. (1955). Cardinal Welfare, Individualistic Ethics, and Interpersonal Comparisons of Utility. *Journal of Political Economy*, *61*, 434–435.

Harsanyi, J. C. (1977). *Rational Behavior and Bargaining Equilibrium in Games and Social Situations*. Cambridge, UK: Cambridge University Press.

Herstein, I. N. & Milnor, J. (1953, April). An Axiomatic Approach to Measurable Utility. *Econometrica*, *21*(2), 291–297. doi:10.2307/1905540

Jones, P. & Sugden, R. (1982). Evaluating choice. *International Review of Law and Economics*, *2*(1), 47–65.

Judd, K. L. (1985, February). The law of large numbers with a continuum of IID random variables. *Journal of Economic Theory*, *35*(1), 19–25. doi:10.1016/0022-0531(85)90059-6

Karni, E. & Weymark, J. A. (1998, May 7). An informationally parsimonious impartial observer theorem. *Social Choice and Welfare*, *15*(3), 321–332. doi:10.1007/s003550050108

Kolm, S.-C. (2010). On real economic freedom. *Social Choice and Welfare*.

Koopmans, T. C. (1964). On flexibility of future preference. In M. W. Shelley & G. L. Bryan (Eds.), *Human Judgments and Optimality*. New York: Wiley.

Kraskov, A., Stögbauer, H., & Grassberger, P. (2004). Estimating mutual information. *Physical review E*, *69*(6), 66138.

Kreps, D. M. (1979, May). A Representation Theorem for "Preference for Flexibility". *Econometrica*, *47*(3), 565. doi:10.2307/1910406. JSTOR: 1910406?origin=crossref

Mailath, G. J., Samuelson, L., & Swinkels, J. M. (1993, March). Extensive Form Reasoning in Normal Form Games. *Econometrica*, *61*(2), 273. doi:10.2307/2951552

Nehring, K. & Puppe, C. (1999). On the multi-preference approach to evaluating opportunities. *Social Choice and Welfare*, *16*, 41–63.

Nehring, K. & Puppe, C. (2008). Diversity and the Metric of Opportunity, 18.

Nehring, K. & Puppe, C. (2009). Diversity. In *Handbook of Rational and Social Choice*. Oxford, UK: Oxford University Press.

Pattanaik, P. (2009). Limits of utilitarianism as the ethical basis of public action. In P. Anand, P. Pattanaik, & C. Puppe (Eds.), *Handbook of rational and social choice*. Oxford University Press.

Pattanaik, P. K. (1994). Rights and freedom in welfare economics. *European Economic Review*, *38*, 731–738.

Pattanaik, P. K. & Xu, Y. (1998). On preference and freedom. *Theory and Decision*, *44*(2), 173–198.

Pattanaik, P. K. & Xu, Y. (1990). On ranking opportunity sets in terms of freedom of choice. *Recherches Economiques de Louvain*, *56*, 383–390.

Peleg, B. (1997). Effectivity functions, game forms, games, and rights. *Social choice and welfare*, *15*(1), 67–80.

Peragine, V. (1999). The distribution and redistribution of opportunity. *Journal of Economic Surveys*, *13*(1), 37–69.

Phibbs, C. F. (2009). *The Montgomery bus boycott: A history and reference guide*. Greenwood.

Puppe, C. & Xu, Y. (2010, October). Essential alternatives and freedom rankings. *Social Choice and Welfare*, *35*(4), 669–685. doi:10.1007/s00355-010-0457-6

Rosenbaum, E. F. (2000). On Measuring Freedom. *Journal of Theoretical Politics*, *12*(2), 205–227.

Ryan, M. (2016, December). Essentiality and convexity in the ranking of opportunity sets. *Social Choice and Welfare*, *47*(4), 853–877. doi:10.1007/s00355-016-0994-8

Safra, Z. & Weissengrin, E. (2003, March 1). Harsanyi's impartial observer theorem with a restricted domain. *Social Choice and Welfare*, *20*(2), 177–187. doi:10.1007/s003550200161

Sen, A. (1988). Freedom of Choice: Concept and Content. *European Economic Review*, *32*, 269–294.

Shannon, C. E. (1948). A Mathematical Theory of Communication. *The Bell System Technical Journal*, *27*(3), 379–423, 623–656.

Sher, I. (2018, July 1). Evaluating Allocations of Freedom. *The Economic Journal*, *128*(612), F65–F94. doi:10.1111/ecoj.12455

Shnayderman, R. (2016, January). Ian Carter's non-evaluative theory of freedom and diversity: A critique. *Social Choice and Welfare*, *46*(1), 39–55. doi:10.1007/s00355-015-0902-7

Sugden, R. (1998). The Metric of Opportunity. *Economics and Philosophy*, *14*, 307–337.

Sugden, R. (2003, July). Opportunity as a Space for Individuality: Its Value and the Impossibility of Measuring It. *Ethics*, *113*(4), 783–809. doi:10.1086/373953

Sugden, R. (2007). The value of opportunities when preferences are unstable. *Social Choice and Welfare*, *29*, 665–682.

Suppes, P. (1996). The nature and measurement of freedom. *Social Choice and Welfare*, *13*(2), 183–200.

Theoharis, J. (2015). *The rebellious life of Mrs. Rosa parks*. Beacon Press.

Thompson, F. B. (1952). Equivalence of games in extensive form. RAND Corporation.

van Hees, M. (2004, February 1). Freedom of choice and diversity of options: Some difficulties. *Social Choice and Welfare*, *22*(1), 253–266. doi:10.1007/s00355-003-0285-z

van Hees, M. & Wissenburg, M. (1999, March). Freedom and Opportunity. *Political Studies*, *47*(1), 67–82. doi:10.1111/1467-9248.00188

Weymark, J. A. (2011). A reconsideration of the Harsanyi-Sen debate on utilitarianism. In *Interpersonal Comparisons of Well-Being*. Cambridge, UK: Cambridge University Press.

Wooding, S., Kim, U.-k., Bamshad, M. J., Larsen, J., Jorde, L. B., & Drayna, D. (2004, April). Natural Selection and Molecular Evolution in PTC, a Bitter-Taste Receptor Gene. *The American Journal of Human Genetics*, *74*(4), 637–646. doi:10.1086/383092

A Proof of Theorem 1

The overall proof structure is as follows:

1. We first prove some technical lemmas that are useful. These include connectedness of the order topology on the set of processes and a result that allows us to deduce a quasi-separable representation of the form $f(x,z) + g(y,z)$ from two conditionally additive representations of the form $h(f(x,z) + g(y,z), z)$. The key axioms used are Continuity and Outcome Equivalence.

2. Next, we show that on the space of processes that are lotteries (i.e., in which no player other than nature is influential), we have an expected utility representation. The key axioms used are Lottery Independence and Outcome Equivalence.

3. Following, we prove that there exists a representation that is quasi-separable across players and their strategies conditional on the outcome probabilities. More specifically, we show that this representation is linear in the probabilities of strategies. The key axioms used are Strategy Independence, Outcome Equivalence and the expected utility representation over lotteries.

4. Having obtained quasi-separability across all players we can focus on processes that have only a single influential player. We show that the preferences over these processes are additively separable across outcomes. All assumptions except Strategy Independence are used in this step.

5. Next, we combine the linear representation across strategies with the additive representation across outcomes.

6. Lastly, we employ the fundamental equation of the theory of information to solve for the procedural component. The key axiom used is Subprocess Monotonicity and the additive separability of the procedural preferences across strategies and from the expected utility of the subprocesses.

To state each Lemma concisely, we omit repeating the axioms employed in the theorem.

A.1 Technical Lemmas

We define the order topology on $\mathcal{P}$ as the topology generated by the intersections of sets of the form $\{p \in \mathcal{P} : p \succ p'\}$ and $\{p \in \mathcal{P} : p'' \succ p\}$ for arbitrary $p', p'' \in \mathcal{P}$.

Lemma 1. *$\mathcal{P}$ is connected in the order topology.*

Proof. By connectedness of the real numbers and Continuity, the order topology on any subspace of $\mathcal{P}$ of the form $\{(G, \theta) : \theta = \alpha\theta' \oplus (1-\alpha)\theta''\}$ is connected. By completeness and transitivity of the relation, this topology is identical to the subspace topology of the order topology on $\mathcal{P}$. If $\mathcal{P}$ is not connected, then it is the union of two nonempty disjoint open sets $\mathcal{P}'$ and $\mathcal{P}''$. Take any element $p' \in \mathcal{P}'$ and $p'' \in \mathcal{P}''$. The order topology on $\mathcal{P}''' = \{(G, \theta) : \theta = \alpha\theta' \oplus (1-\alpha)\theta''\}$ is disconnected by the nonempty open sets $\mathcal{P}' \cap \mathcal{P}'''$ and $\mathcal{P}'' \cap \mathcal{P}'''$, yielding a contradiction. Thus, $\mathcal{P}$ is connected. $\square$

Lemma 2. *Suppose $f(g(x, y), z) = xa(y, z) + b(y, z)$ holds for continuous functions $f : \mathbb{R} \times \mathcal{Z} \to \mathbb{R}$ and $g : \mathbb{R} \times \mathcal{Y} \to \mathbb{R}$ on some sets $\mathcal{Y}$, $\mathcal{Z}$. Let f and g be invertible in the first argument, then,*

$$\begin{aligned} f(r, z) &= h^{-1}(r)j(z) + k(z) \\ g(x, y) &= h(xl(y) + m(y)). \end{aligned} \tag{29}$$

Proof. Let f^{-1}, g^{-1} denote the inverses of f and g in their first arguments, respectively. We use invertibility of the two functions to derive:

$$\begin{aligned} g(x, y) =& f^{-1}(xa(y, z_0) + b(y, z_0), z_0) \\ f(r, z) =& g^{-1}(r, y_0)a(y_0, z) + b(y_0, z) \\ f(g(x, y), z) =& g^{-1}(g(x, y), y_0)a(y_0, z) + b(y_0, z) \\ =& g^{-1}(f^{-1}(xa(y, z_0) + b(y, z_0), z_0), y_0)a(y_0, z) + b(y_0, z) \end{aligned} \tag{30}$$

which, by the assumption that f and g are continuous, is affine in x if and only if $g^{-1}(r, y_0)$ and $f^{-1}(r, z_0)$ are (up to an affine transformation) inverses to each other. The result then follows by appropriate definitions for h, j, k, l, m. $\square$

A.2 Expected Utility Representation on Lotteries

Lemma 3 (Expected Utility on Lotteries)**.** *On the set of nature's lotteries over outcomes, $\{(G, \theta) : (G, \theta) = D_{\mathcal{N}}(G, \theta)\}$, the relation $\succsim$ can be represented by*

$U[G,\theta] = \sum_{o\in\mathcal{O}} \rho[o]U[o]$.

Remark 3. $U[x]$ is shorthand for $U[(\mathcal{N}, \{a\}, a \mapsto \mathbb{1}_x), \mathbb{1}_{\mathbb{1}_a}]$.

Proof. We show that $\succsim$ fulfills the assumptions of Herstein and Milnor (1953). By Outcome Equivalence,

$$D_{\mathcal{N}}(G,\theta) = (G,\theta) \Rightarrow (G,\theta) \approx ((\mathcal{N}, \{a\}, a \mapsto \rho_{G,\theta}), \mathbb{1}_{\mathbb{1}_a}) \tag{31}$$

That is, any process in which no player is influential is outcome equivalent to a trivial process with a single action profile a in which a lottery over outcomes is resolved with probabilities $\rho_{G,\theta}$. The set of probability distributions form a mixture space. Furthermore,

$$\alpha(G,\theta) \oplus (1-\alpha)(G',\theta') \approx ((\mathcal{N}, \{a\}, a \mapsto \alpha\rho_{G,\theta} \oplus (1-\alpha)\rho_{G',\theta'}), \mathbb{1}_{\mathbb{1}_a}) \tag{32}$$

and therefore mixtures between processes translate into mixtures between outcome probability distributions. It follows from Lottery Independence that:

$$\begin{array}{cccc}
 & ((\mathcal{N}, \{a\}, a \mapsto \rho_{G,\theta}), \mathbb{1}_{\mathbb{1}_a}) & \succsim & ((\mathcal{N}, \{a\}, a \mapsto \rho_{G',\theta'}), \mathbb{1}_{\mathbb{1}_a}) \\
 & \approx & & \approx \\
 & (G,\theta) & \succsim & (G',\theta') \\
 & = & & = \\
 & D_{\mathcal{N}}(G,\theta) & \succsim & D_{\mathcal{N}}(G',\theta') \\
\Leftrightarrow & \alpha D_{\mathcal{N}}(G,\theta) \oplus (1-\alpha) D_{\mathcal{N}}(G'',\theta'') & \succsim & \alpha D_{\mathcal{N}}(G',\theta') \oplus (1-\alpha) D_{\mathcal{N}}(G'',\theta'') \\
 & \approx & & \approx \\
 & ((\mathcal{N}, \{a\}, a \mapsto \alpha\rho_{G,\theta} \oplus (1-\alpha)\rho_{G'',\theta''}), \mathbb{1}_{\mathbb{1}_a}) & \succsim & ((\mathcal{N}, \{a\}, a \mapsto \alpha\rho_{G',\theta'} \oplus (1-\alpha)\rho_{G'',\theta''}), \mathbb{1}_{\mathbb{1}_a})
\end{array} \tag{33}$$

It follows that on the set of lotteries, $\succsim$ fulfills the independence axiom (Herstein & Milnor, 1953, Axiom 3) with respect to the outcome probabilities $\rho_{G,\theta}$. Rationality and Continuity guarantee their Axioms 1 and 2. The existence of an expected utility representation follows from their Theorem 8. □

A.3 Conditional Linearity in Probabilities of Strategies

Lemma 4 (Separability in Strategies)**.** *There exists a representation of the form:*

$$U[G,\theta] = \sum_{i\in\mathcal{N}} \sum_{\mu_i} \theta_i[\mu_i] v_i(\rho_{G,\theta}|\mu_i, \rho_{G,\theta}) \equiv \sum_{i\in\mathcal{N}} U_i[G,\theta]. \tag{34}$$

Proof. Both the information over strategies $\theta = \prod_{i \in \mathcal{N}} \theta_i$ and the information over strategies of particular individuals θ_i are probability distributions and therefore elements of mixture spaces. We first use Strategy Independence to derive a conditional expected utility representation for each θ_i. We derive the following conditional independence property. If for all $j \in \mathcal{N} - \{i\}$ and all $\mu_j \in \Delta A_j$:

$$\rho_{G,\theta_i \otimes \theta_{-i}} | \mu_j = \rho_{G,\theta'_i \otimes \theta'_{-i}} | \mu_j = \rho_{G,\theta''_i \otimes \theta''_{-i}} | \mu_j \tag{35}$$

$$\rho_{G,\theta_i \otimes \theta_{-i}} = \rho_{G,\theta'_i \otimes \theta'_{-i}} = \rho_{G,\theta''_i \otimes \theta''_{-i}} \tag{36}$$

then

$$(G, (\frac{1}{2}\theta_i \oplus \frac{1}{2}\theta''_i) \otimes \theta_{-i}) \succsim (G, (\frac{1}{2}\theta'_i \oplus \frac{1}{2}\theta''_i) \otimes \theta_{-i}) \tag{37}$$

$$\Leftrightarrow \quad (G, \theta_i \otimes \theta_{-i}) \succsim (G, \theta'_i \otimes \theta_{-i}) \tag{38}$$

We emphasize at this point that the mixture in the above processes each represents uncertainty of the policy maker about the strategies played by the player, not a random choice by nature. The proof of the above independence result uses Strategy Independence and Outcome Equivalence. First we remove choice over strategies in $\mathcal{M}_i = supp[\theta''_i]$. Note that for this purpose, we may assume that θ''_i has disjoint support from θ_i and θ'_i since by Continuity we can choose a disjoint support that is arbitrarily close to the actual support of θ''_i. Applying the choice removal $D^{\mathcal{M}_i}$ on both sides leaves the preference unchanged. Using an outcome equivalent transformation, we have:

$$(G, (\frac{1}{2}\theta_i \oplus \frac{1}{2}\theta''_i) \otimes \theta_{-i}) \succsim (G, (\frac{1}{2}\theta'_i \oplus \frac{1}{2}\theta''_i) \otimes \theta_{-i}) \tag{39}$$

$$\Leftrightarrow \quad (G, (\frac{1}{2}\theta_i \oplus \frac{1}{2}\mathbb{1}_{\mu_i^*}) \otimes \theta_{-i}) \succsim (G, (\frac{1}{2}\theta'_i \oplus \frac{1}{2}\mathbb{1}_{\mu_i^*}) \otimes \theta_{-i}) \tag{40}$$

where $\mathbb{1}_{\mu_i^*}$ denotes a mixed strategy in which i plays the actions with the same probability with which they are played in θ''_i. Now suppose the marginal distributions fulfill (35). Then there exists some outcome equivalent transformation such that

$$(G, (\frac{1}{2}\theta_i \oplus \frac{1}{2}\mathbb{1}_{\mu_i^*}) \otimes \theta_{-i}) \succsim (G, (\frac{1}{2}\theta'_i \oplus \frac{1}{2}\mathbb{1}_{\mu_i^*}) \otimes \theta_{-i}) \tag{41}$$

$$\Leftrightarrow \quad (G, (\frac{1}{2}\theta_i \oplus \frac{1}{2}\mathbb{1}_{\mu_i^{**}}) \otimes \theta_{-i}) \succsim (G, (\frac{1}{2}\theta'_i \oplus \frac{1}{2}\mathbb{1}_{\mu_i^{**}}) \otimes \theta_{-i}) \tag{42}$$

where $\mathbb{1}_{\mu_i^{**}}$ plays each action with the same probability as the marginal probability in θ_i. It follows that:

$$(G, (\frac{1}{2}\theta_i \oplus \frac{1}{2}\mathbb{1}_{\mu_i^{**}}) \otimes \theta_{-i}) \succsim (G, (\frac{1}{2}\theta_i' \oplus \frac{1}{2}\mathbb{1}_{\mu_i^{**}}) \otimes \theta_{-i}) \tag{43}$$
$$\Leftrightarrow \quad (G, \theta_i \otimes \theta_{-i}) \succsim (G, (\frac{1}{2}\theta_i' \oplus \frac{1}{2}\theta_i) \otimes \theta_{-i}) \tag{44}$$

Proceeding in a similar manner we can derive

$$(G, (\frac{1}{2}\theta_i \oplus \frac{1}{2}\mathbb{1}_{\mu_i^{***}}) \otimes \theta_{-i}) \succsim (G, (\frac{1}{2}\theta_i' \oplus \frac{1}{2}\mathbb{1}_{\mu_i^{***}}) \otimes \theta_{-i}) \tag{45}$$
$$\Leftrightarrow \quad (G, (\frac{1}{2}\theta_i \oplus \frac{1}{2}\theta_i') \otimes \theta_{-i}) \succsim (G, \theta_i' \otimes \theta_{-i}) \tag{46}$$

where $\mathbb{1}_{\mu_i^{***}}$ plays each action with the same probability as in θ_i'.

Combining (44) with (46), we have by transitivity the desired result (37). By Rationality and Continuity, using Theorem 8 of Herstein and Milnor (1953), it follows that for fixed conditional outcome probabilities given the other player's strategies, we have an expected utility representation on θ_i.[19]

Next, it holds for $\theta_{\mathcal{N}'} \otimes \theta_{\mathcal{N}-\mathcal{N}'} = \prod_{i\in\mathcal{N}'} \theta_i \otimes \prod_{i\in\mathcal{N}-\mathcal{N}'} \theta_i$ that if for all $j \in \mathcal{N} - \mathcal{N}'$ and all $\mu_j \in \Delta A_j$:

$$\rho_{G,\theta_{\mathcal{N}'}\otimes\theta_{\mathcal{N}-\mathcal{N}'}}|\mu_j = \rho_{G,\theta'_{\mathcal{N}'}\otimes\theta_{\mathcal{N}-\mathcal{N}'}}|\mu_j \tag{47}$$
$$= \rho_{G,\theta_{\mathcal{N}'}\otimes\theta'_{\mathcal{N}-\mathcal{N}'}}|\mu_j \tag{48}$$
$$= \rho_{G,\theta'_{\mathcal{N}'}\otimes\theta'_{\mathcal{N}-\mathcal{N}'}}|\mu_j, \tag{49}$$
$$\rho_{G,\theta_{\mathcal{N}'}\otimes\theta_{\mathcal{N}-\mathcal{N}'}} = \rho_{G,\theta'_{\mathcal{N}'}\otimes\theta_{\mathcal{N}-\mathcal{N}'}} = \rho_{G,\theta_{\mathcal{N}'}\otimes\theta'_{\mathcal{N}-\mathcal{N}'}} = \rho_{G,\theta'_{\mathcal{N}'}\otimes\theta'_{\mathcal{N}-\mathcal{N}'}}, \tag{50}$$

then:

$$(G, \theta_{\mathcal{N}'} \otimes \theta_{\mathcal{N}-\mathcal{N}'}) \succsim (G, \theta'_{\mathcal{N}'} \otimes \theta_{\mathcal{N}-\mathcal{N}'}) \tag{51}$$
$$\Leftrightarrow (G, \theta_{\mathcal{N}'} \otimes \theta'_{\mathcal{N}-\mathcal{N}'}) \succsim (G, \theta'_{\mathcal{N}'} \otimes \theta'_{\mathcal{N}-\mathcal{N}'}) \tag{52}$$

The proof is almost identical to the above, except that instead of removing choice over the strategies in $\mathcal{M}_i$ for a single player i, instead the choice over the entire strategies ΔA_j is removed for all $j \in \mathcal{N} - \mathcal{N}'$. We therefore have an additively separable representation across the probabilities of the strategies of each of the players.

Finally, we show that the expected utility representation and the

[19] Although this result holds for arbitrary i, this of course does not yet imply that (for fixed probability of o) the aggregation (across i) of the expected utility representations is additive.

additive representation across players are jointly additive. If for some individuals i, j,

$$\rho_{G,\theta_{ij}\otimes\theta_{-ij}}|\mu_k = \rho_{G,\theta'_{ij}\otimes\theta_{-ij}}|\mu_k \tag{53}$$
$$= \rho_{G,\theta_{ij}\otimes\theta'_{-ij}}|\mu_k \tag{54}$$
$$= \rho_{G,\theta'_{ij}\otimes\theta_{-ij}}|\mu_k, \quad \mu_k \in \Delta A_k, \forall k \in \mathcal{N} - \{i, j\}, \tag{55}$$
$$\rho_{G,\theta_{ij}\otimes\theta_{-ij}} = \rho_{G,\theta'_{ij}\otimes\theta_{-ij}} = \rho_{G,\theta_{ij}\otimes\theta'_{-ij}} = \rho_{G,\theta'_{ij}\otimes\theta'_{-ij}} \tag{56}$$

then,

$$(G, \frac{1}{2}\theta_i \oplus \frac{1}{2}\theta'_i \otimes \frac{1}{2}\theta_j \oplus \frac{1}{2}\theta'_j \otimes \theta_{-ij}) \succsim (G, \frac{1}{2}\theta_i \oplus \frac{1}{2}\theta'_i \otimes \frac{1}{2}\theta_j \oplus \frac{1}{2}\theta'_j \otimes \theta_{-ij}) \tag{57}$$
$$\Leftrightarrow (G, \frac{1}{2}\theta_i \oplus \frac{1}{2}\theta''_i \otimes \frac{1}{2}\theta_j \oplus \frac{1}{2}\theta''_j \otimes \theta_{-ij}) \succsim (G, \frac{1}{2}\theta_i \oplus \frac{1}{2}\theta''_i \otimes \frac{1}{2}\theta_j \oplus \frac{1}{2}\theta''_j \otimes \theta_{-ij}) \tag{58}$$

When fixing all θ_k, ρ, and all $\rho|\mu_k$ for all individuals $k \neq i, j$ and strategies μ_k in the support, we can therefore find an additive representation of the form:

$$U[G, \frac{1}{2}\theta_i \oplus \frac{1}{2}\theta'_i \otimes \frac{1}{2}\theta_j \oplus \frac{1}{2}\theta'_j \otimes \theta_{-ij}] \tag{59}$$
$$= f_i[\theta_i] + g_i[\theta'_i] + f_j[\theta_j] + g_j[\theta'_j] \tag{60}$$

Since for the probabilities of strategic choice of each individual we have an expected utility representation, we indeed have:

$$U[G, \alpha\theta_i \oplus (1-\alpha)\theta'_i \otimes \beta\theta_j \oplus (1-\beta)\theta'_j \otimes \theta_{-ij}] \tag{61}$$
$$= \alpha h_i[\theta_i] + (1-\alpha)h_i[\theta'_i] + \beta h_j[\theta_j] + (1-\beta)h_j[\theta'_j] \tag{62}$$

since the expected utility representation is additive and uniqueness of additive representations applies. Further, we can derive $h_i[\theta_i] = \sum_{\mu_i} \theta_i[\mu_i]w[\mu_i]$ using Cauchy's functional equation. By Outcome Equivalence, increasing the probability that μ_i will be played instead of μ'_i only matter if $\rho_{G,\theta}|\mu_i \neq \rho_{G,\theta}|\mu'_i$, thus: $h_i[\theta_i] = \sum_{\mu_i} \theta_i[\mu_i]v_i[\rho_{G,\theta}|\mu_i]]$. While we have only shown additive separability of the expected utility representations for i and j, the extension to n individuals is straightforward and we therefore obtain a representation:

$$U[G,\theta] = V\left[\sum_i \sum_{\mu_i} \theta_i[\mu_i] v_i[\rho_{G,\theta}|\mu_i], \rho_{G,\theta}], \rho_{G,\theta}\right] \tag{63}$$

for arbitrary $\rho_{G,\theta}$.

What is left to show is that V is affine in its first argument. For this, we assume there are three influential players.[20] Consider a process θ such that:

$$\theta = \theta_{\{\mathcal{B},\mathcal{C},\mathcal{D}\}} \otimes \theta_{\mathcal{B}} \otimes \theta_{\mathcal{C}} \otimes \theta_{\mathcal{D}} \tag{64}$$
$$\theta_{\{\mathcal{B},\mathcal{C},\mathcal{D}\}} = \theta_1 \otimes \prod_{j\neq 1} \mathbb{1}_{d_j} \tag{65}$$
$$\theta_{\mathcal{B}} = \theta_2 \otimes \prod_{j\neq 2} \mathbb{1}_{b_j} \tag{66}$$
$$\theta_{\mathcal{C}} = \theta_3 \otimes \prod_{j\neq 3} \mathbb{1}_{c_j} \tag{67}$$
$$\theta_{\mathcal{D}} = \theta_4 \otimes \prod_{j\neq 4} \mathbb{1}_{d_j} \tag{68}$$

This process has three subprocesses in which on each subprocess only a single player chooses between strategies. All other players play a single pure strategy. Player 1's strategies determine which of the subprocesses is being played. We assume that the three subprocesses each have two disjoint outcomes from the remainder of the game form. We have the representation:

$$V\Bigg[\sum_{\mu_1} \theta_1[\mu_1] v_1[\rho_{G,\theta}|\mu_1, \rho_{G,\theta}] \tag{69}$$
$$+ \sum_{\mu_2} \theta_2[\mu_2] v_2[\rho_{G,\theta}|\mu_2, \rho_{G,\theta}] \tag{70}$$
$$+ \sum_{\mu_3} \theta_3[\mu_3] v_3[\rho_{G,\theta}|\mu_3, \rho_{G,\theta}] \tag{71}$$
$$+ \sum_{\mu_4} \theta_4[\mu_4] v_4[\rho_{G,\theta}|\mu_4, \rho_{G,\theta}], \rho_{G,\theta}\Bigg] \tag{72}$$

For fixed outcome probabilities, this representation is additively separable in the three subprocesses. Note that by Subprocess Monotonicity, on

[20] It is straightforward to adapt the proof to a single influential player. Employing distinct influential players for each subprocess makes the proof notationally clearer, however.

the space of processes of the above form, $\succsim$ fulfills joint independence across subprocesses for fixed θ_1. By Gorman (1968) there then exists an additively separable representation of the form:

$$W[f_2[\theta_2,\theta_1] + f_3[\theta_3,\theta_1] + f_4[\theta_4,\theta_1],\theta_1] \tag{73}$$

where W is monotone and thus invertible in its first argument. We show that W must be affine. Indeed, by the existence of an expected utility representation in case $\theta_1 = \mathbb{1}_{\mu_1},\ldots,\theta_4 = \mathbb{1}_{\mu_4}$ it follows that W is affine if $\theta_1 = \mathbb{1}_{\mu_1}$. Without loss of generality then,

$$\begin{aligned}
V\Bigg[& v_1[\rho_{G,\theta}|\mu_1,\rho_{G,\theta}] && (74)\\
& + \sum_{\mu_2}\theta_2[\mu_2]v_2[\rho_{G,\theta}|\mu_2,\rho_{G,\theta}] && (75)\\
& + \sum_{\mu_3}\theta_3[\mu_3]v_3[\rho_{G,\theta}|\mu_3,\rho_{G,\theta}] && (76)\\
& + \sum_{\mu_4}\theta_4[\mu_4]v_4[\rho_{G,\theta}|\mu_4,\rho_{G,\theta}],\rho_{G,\theta}\Bigg] = f_2[\theta_2,\theta_1] + f_3[\theta_3,\theta_1] + f_4[\theta_4,\theta_1] && (77)
\end{aligned}$$

if $\theta_1 = \mathbb{1}_{\mu_1}$. Since the three players 2, 3, 4 are influential, their sum components on the LHS vary for fixed $\rho_{G,\theta}$. It follows that for fixed $\rho_{G,\theta}$, both the RHS and the first argument of V are additive representations across $\theta_2,\ldots,\theta_4$. We therefore obtain by uniqueness of additive representations that V is affine in the first argument. Since $\sum_{\mu_i}\theta_i[\mu_i] = 1$, it is without loss of generality to assume that V is the unit transformation. We therefore obtain the desired representation:

$$\begin{aligned}
U[G,\theta] &= \sum_{i\in\mathcal{N}}\sum_{\mu_i}\theta[\mu_i]v_i[\rho_{G,\theta}|\mu_i,\rho_{G,\theta}] && (78)\\
&= \sum_{i\in\mathcal{N}} U_i[G,\theta] && (79)
\end{aligned}$$

$\square$

A.4 Additive Separability on Subprocesses

The remainder of the proof is about specifying the functional form of U_i. In order to identify v_i for some player i, we need to consider only processes in which a single player i is influential, since

$$
\begin{aligned}
& U_i[G,\theta] && (80)\\
=& U[D_{-i}(G,\theta)] - \sum_{j\neq i} U_j[D_{-i}(G,\theta)] && (81)\\
=& U[D_{-i}(G,\theta)] - \sum_{j\neq i} U[D_{-j}D_{-i}(G,\theta)] + \sum_{j\neq i}\sum_{k\neq j} U_k[D_{-j}D_{-i}(G,\theta)] && (82)\\
=& U[D_{-i}(G,\theta)] - U[D_{\mathcal{N}}(G,\theta)] + U_i[D_{\mathcal{N}}(G,\theta)]. && (83)
\end{aligned}
$$

The latter two terms of the expressions have already been determined as expected utility representations. We therefore focus on processes of the form $D_{-i}(G,\theta)$. Using Outcome Equivalence, we can further focus on the processes in which all uncertainty is resolved in the mixed strategies of player i instead of by nature. For this, we define a game form $G^* = (\mathcal{N}, \mathcal{A}^*, o^*)$ such that for some bijection $f : \mathcal{O} \to \mathcal{A}^*$, $o[f(x)] = \mathbb{1}_x$.

Lemma 5 (Equivalence to game form without nature)**.** *Suppose that* $\theta[\mu] = \theta^*[f\#\bigoplus_a \mu[a]o[a]]$ *for all* $\mu \in supp[\theta]$*, then* $(G,\theta) \sim (G^*,\theta^*)$*.*

Proof. The two processes are outcome equivalent. □

Lemma 6 (Power Removal Lemma)**.** *Let* (G,θ) *be in outcome form with mapping* $f : \mathcal{O} \to \mathcal{A}^*$*. Let* $g : \mathcal{O} \to \prod_i \mathcal{O}_i$ *be the product mapping of each of the canonical maps* g_i *of the partition* $\mathcal{O}_i$*. Let* $\mathcal{A}' = \{\mathcal{B} \subseteq \mathcal{A} : \exists x_i \in \mathcal{O}_i : f[x_i] = \mathcal{B}\}$*. Let* θ' *and* o' *fulfill for all* $a' \in \mathcal{A}'$ *and all* μ*:*

$$o'[a'] = \mathbb{1}_{f^{-1}[a']} \otimes \prod_{j\neq i} g_j\#\rho_{G,\theta} \tag{84}$$

$$\theta'[f \circ g_i \circ f^{-1}\#\mu] = \theta[\mu] \tag{85}$$

Then $(G,\theta) \sim (G',\theta')$*.*

Thus, we replace each action profile in $\mathcal{A}$ with an action that yields a particular outcome from $\mathcal{O}_i$ and the same lottery across all other outcomes. This naturally generates an outcome equivalent process.

Proof. We show that the processes are outcome equivalent. For all $j \neq i$ and all $a' \in \mathcal{A}'$,, $\mu_i \in \Delta\mathcal{A}_i$: $g_j\#\rho_{G,\theta} = g_j\#o'[a'] = g_j\#\rho_{G',\theta'}|\mu_i = g_j\#\rho_{G',\theta'}$. For player i, we first note that $f \circ g_i \circ f^{-1}$ is the canonical mapping from $\mathcal{A}$ to $\mathcal{A}'$, since $\mathcal{A}'$ is a partition of $\mathcal{A}$. All actions in a' yield the same outcome for i as each $a \in a'$. Therefore, $g_i\#\rho_{G,\theta}|\mu = g_i\#\rho_{G',\theta'}|f \circ g_i \circ f^{-1}\#\mu$. □

Note that in processes in which power has been removed, there exists a bijection $f' : \mathcal{O}_i \to \mathcal{A}$ such that the action a of player i determines the outcome $f^{-1}[a]$ with certainty.

To use Subprocess Monotonicity, we require disjoint subgames. Consider the canonical mapping $g : \mathcal{O} \to \mathcal{O}_i$ that maps each element to their equivalence class. Note that the image $f[g^{-1}[x_i]]$ is a subgame of G^*. We now create for every process (G^*, θ) an indifferent process (G^*, θ^*) such that $\mu[f[x]] = \mu[f[x_i]] \frac{\bigoplus_{\mu'} \theta[\mu']\mu'[x]}{\bigoplus_{\mu'} \theta[\mu']\mu'[x_i]}$ for all $\mu \in supp[\theta^*]$. In other words, while strategies may differ about the probabilities of elements of $\mathcal{O}_i$, conditional on reaching $x_i \in \mathcal{O}_i$, all strategies yield the same probability distribution over outcomes. This means that player i can only influence the own outcomes. To prove this, we must first find a way to decompose a process into subgames. This is done in the following Lemma.

Lemma 7 (Disjoint Subprocess Decomposition Lemma)**.** *Let (G, θ) be in powerless outcome form with mapping $f : \mathcal{O} \to \mathcal{A}$. Consider a partition $\overline{\mathcal{A}}$ of $\mathcal{A}$. For any $\mathcal{B} \in \overline{\mathcal{A}}$, denote by $\theta_{\mathcal{B}}$ the conditional probabilities fulfilling $(G, \theta_{\mathcal{B}}) = (G, \theta)|\mathcal{B}$. Let the function $h : supp[\theta] \times \prod_{\mathcal{B} \in \overline{\mathcal{B}}} supp_{\theta_{\mathcal{B}}} \to \Delta\mathcal{A}$ map strategies on every subgame to strategies in the game such that: Define,*

$$h[k\#\mu, \{\mu_{\mathcal{B}}\}_{\mathcal{B}}][\mathcal{C}] = \sum_{\mathcal{B}} k\#\mu[\mathcal{B}]\mu_{\mathcal{B}}[\mathcal{B} \cap \mathcal{C}] \tag{86}$$

$$\theta^*[h[\mu', \{\mu_{\mathcal{B}}\}]] = \theta[\mu'] \prod_{\mathcal{B}} \theta_{\mathcal{B}}[\mu_{\mathcal{B}}] \tag{87}$$

$$b : \left(k\#\mu, \{\mu|\mathcal{B}\}_{\mathcal{B} \in \overline{\mathcal{A}}}\right) \mapsto \bigoplus_{\mathcal{B}} k\#\mu[\mathcal{B}]\mu|\mathcal{B} \tag{88}$$

$$\theta_{\overline{\mathcal{A}}}[k\#\mu] = \theta[\mu] \tag{89}$$

$$\theta^* = b\# \left(\theta_{\overline{\mathcal{A}}} \otimes \prod_{\mathcal{B}} \theta_{\mathcal{B}}\right). \tag{90}$$

Then, $(G^*, \theta) \sim (G^*, \theta^*)$.

Proof. The two processes are identical on every subprocess. By subprocess monotonicity, if after equalizing the outcomes within each of the subprocesses the two processes are equivalent, then they are indifferent. Note that after identifying the outcomes of the actions within each subprocess, all strategies μ that are identical under the pushforward $k\#\mu$ are strategically equivalent. It follows that in this case every $\mu \in supp_i[\theta]$ is strategically equivalent to all elements of $h[\mu, \prod_{\mathcal{B} \in \overline{\mathcal{B}}} supp_{\theta_{\mathcal{B}}}]$. Since $\theta^*[h[\mu, \prod_{\mathcal{B} \in \overline{\mathcal{B}}} supp_{\theta_{\mathcal{B}}}]] = \theta[\mu]$, it follows that the processes are equivalent. □

We now focus on processes in outcome form in which i has no power and with some subprocess decomposition corresponding to a partition $\overline{\mathcal{O}}_i$ of $\mathcal{O}_i$. These processes have all information stripped that are irrelevant for the determination of U_i. Note that the bijection f, every partition of $\mathcal{O}_i$ corresponds to a unique partition of $\mathcal{A}$ and vice versa. The specification of a partition of outcomes gives us a unique subprocess decomposition. In the following, we will refer to such processes alternatively as $\overline{\mathcal{O}}_i$-decomposed or $\overline{\mathcal{A}}$-decomposed processes.

Lemma 8. *Let $\mathcal{S}_i$ contain all processes fulfilling $(G, \theta) = D_{-i}(G, \theta)$ in powerless outcome form with bijection $f : \mathcal{O}_i \to \mathcal{A}$. Let $\overline{\mathcal{O}}_i$ be a partition of $\mathcal{O}_i$ containing three elements with jointly essential outcomes. Then $\overline{\mathcal{A}} = \{f[x] : x \in \mathcal{O}_i\}$ there exists a representation $U : \mathcal{S}_i \to \mathbb{R}$ of $\succsim$ such that:*

$$U[G, \theta] = K\left[\sum_{\mathcal{B}\in\overline{\mathcal{A}}} F_{\mathcal{B}}[U[(G, \theta)|\mathcal{B}], \theta^*_{\overline{\mathcal{A}}}], \theta^*_{\overline{\mathcal{A}}}\right]. \tag{91}$$

Proof. We proceed in the following steps:

1. We provide the representation for an arbitrary fixed partition of outcomes.

2. We show that the choice of the partitioning does not influence the representation.

Consider the decomposition $\overline{\mathcal{A}} = \{\mathcal{B}, \mathcal{C}, \mathcal{D}\}$. Without loss of generality, by the previous lemmas we find the indifferent process $(G^*, \theta^*_{\overline{\mathcal{A}}} \otimes \theta^*_{\mathcal{B}} \otimes \theta^*_{\mathcal{C}} \otimes \theta^*_{\mathcal{D}})$.

We use the result of Gorman, 1968 to obtain an additively separable representation. To apply this result, we require a product space, topological connectedness of each dimension of the product space, and continuous preorders on the subsets of dimensions of the product space that are additively separable in the representation. Clearly, the set of processes of the form $(G^*, \theta^*_{\overline{\mathcal{A}}} \otimes \theta^*_{\mathcal{B}} \otimes \theta^*_{\mathcal{C}} \otimes \theta^*_{\mathcal{D}})$ forms a product space with the dimensions being the strategies over subprocesses and the set of strategies $\theta^*_{\{\mathcal{B},\mathcal{C},\mathcal{D}\}}$ that determine which subprocess is played. We use the preorder topology on these subsets which guarantees connectedness by Lemma 1. By Subprocess Monotonicity, $(G^*, \theta_{\mathcal{B}}) \succsim_i (G^*, \theta'_{\mathcal{B}})$ if and only if $(G^*, \theta_{\overline{\mathcal{A}}} \otimes \theta_{\mathcal{B}} \otimes \theta_{\mathcal{C}} \otimes \theta_{\mathcal{D}}) \succsim_i (G^*, \theta_{\overline{\mathcal{A}}} \otimes \theta'_{\mathcal{B}} \otimes \theta_{\mathcal{C}} \otimes \theta_{\mathcal{D}})$. This yields so-called coordinate independence in each of the subprocesses but not joint independence of the three dimensions. To obtain joint independence, we also need a preorder on combinations of subprocesses, for example $\theta_{\mathcal{B}} \otimes \theta_{\mathcal{C}}$. We obtain this preorder by finding for every process the indifferent process $(G^*, \theta_{\{\mathcal{B}\cup\mathcal{C},\mathcal{D}\}} \otimes \theta_{\mathcal{B}\cup\mathcal{C}} \otimes \theta_{\mathcal{D}})$. For such processes, we have that $(G^*, \theta_{\mathcal{B}\cup\mathcal{C}}) \succsim_i (G^*, \theta_{\mathcal{B}\cup\mathcal{C}})$ if and only if $(G^*, \theta_{\{\mathcal{B}\cup\mathcal{C},\mathcal{D}\}} \otimes \theta_{\mathcal{B}\cup\mathcal{C}} \otimes \theta_{\mathcal{D}}) \succsim_i (G^*, \theta_{\{\mathcal{B}\cup\mathcal{C},\mathcal{D}\}} \otimes \theta_{\mathcal{B}\cup\mathcal{C}} \otimes \theta_{\mathcal{D}})$. This yields a well defined preorder on combinations of subprocesses and therefore we have joint independence. From Gorman, 1968 then follows the existence of a representation of the form:

$$U[G,\theta] = K\left[F_{\mathcal{B}}[\theta^*_{\{\mathcal{B},\mathcal{C},\mathcal{D}\}}, \theta^*_{\mathcal{B}}] + F_{\mathcal{C}}[\theta^*_{\{\mathcal{B},\mathcal{C},\mathcal{D}\}}, \theta^*_{\mathcal{C}}] + F_{\mathcal{D}}[\theta^*_{\{\mathcal{B},\mathcal{C},\mathcal{D}\}}, \theta^*_{\mathcal{D}}], \theta^*_{\{\mathcal{B},\mathcal{C},\mathcal{D}\}}\right] \tag{92}$$

The extension to arbitrary finite dimensions follows from a simple induction argument, since for a process in powerless outcome form, the union of any two disjoint subgames form a disjoint subgame. Finally, we note that $F_{\mathcal{B}}$ must be an increasing function of $U[(G,\theta)|\mathcal{B}]$ for changes in $\theta^*_{\mathcal{B}}$. □

Lemma 9. *Suppose $(G,\theta) = D_{-i}(G,\theta)$. Let $\overline{\mathcal{A}}$ be a partition of $\mathcal{A}$ into subgames with disjoint, essential outcomes. Then there exists a representation of $\succsim_i$ in the form*

$$U[G,\theta] = \sum_{\mathcal{B}\in\overline{\mathcal{A}}} U[G,\theta^*_{\mathcal{B}}]M_{\mathcal{B}}[\theta^*_{\overline{\mathcal{A}}}] + L_{\mathcal{B}}[\theta^*_{\overline{\mathcal{A}}}]. \tag{93}$$

The result follows from repeated application of a uniqueness argument of additive representations; if a relation can be represented by a sum of two or more additive functions, then any other such representation must be an affine transformation of this representation.

Proof. By the previous Lemma, we may construct additive representations over processes with decompositions $\overline{\mathcal{A}} = \{\mathcal{B}, \{\mathcal{C},\mathcal{D}\}, \mathcal{E}\}$ and $\overline{\mathcal{A}}' = \{\mathcal{B}, \mathcal{C}, \{\mathcal{D},\mathcal{E}\}\}$. Note that on the set of processes in which $\mathcal{E}$ is null, the representations must agree up to a positive monotone transformation. We assume this transformation to be the identity (this is without loss of generality as we can simply redefine K for one of the representations). Letting $\theta^*_{\overline{A}}$ converge to the case where $\mathcal{E}$ is null yields by Continuity two representations of the form:

$$\begin{aligned} U[G,\theta] =& K[F_{\mathcal{B}}[U[G,\theta^*_{\mathcal{B}}], \theta^*_{\overline{\mathcal{A}}}] + F_{\mathcal{C}\cup\mathcal{D}}[U[G,\theta^*_{\mathcal{C}\cup\mathcal{D}}], \theta^*_{\mathcal{B},\mathcal{C}\cup\mathcal{D}}], \theta^*_{\overline{\mathcal{A}}}] \\ =& K'[F'_{\mathcal{B}}[U[G,\theta^*_{\mathcal{B}}], \theta^*_{\overline{\mathcal{A}}'}] + F'_{\mathcal{C}}[U[G,\theta^*_{\mathcal{C}}], \theta^*_{\overline{\mathcal{A}}'}] + F'_{\mathcal{D}}[U[G,\theta^*_{\mathcal{D}}], \theta^*_{\overline{\mathcal{A}}'}], \theta^*_{\overline{\mathcal{A}}'}] \end{aligned} \tag{94}$$

Since the first argument of K and K' are both additive representations over $\theta^*_{\mathcal{B}}$ and $\theta^*_{\mathcal{C}}$, for fixed $\theta^*_{\{\mathcal{B},\mathcal{C}\cup\mathcal{D}\}}$, the transformation $K^{-1}[K'[\cdot, \theta^*_{\overline{\mathcal{A}}'}], \theta^*_{\overline{\mathcal{A}}}]$

must be affine. We note that all information in $\theta^*_{\overline{\mathcal{A}}}$ is contained in $\theta^*_{\overline{\mathcal{A}'}}$. It follows that

$$F'_{\mathcal{B}}[U[G,\theta^*_{\mathcal{B}}],\theta^*_{\overline{\mathcal{A}'}}] = F_{\mathcal{B}}[U[G,\theta^*_{\mathcal{B}}],\theta^*_{\overline{\mathcal{A}}}]M_{\mathcal{B}}[\theta^*_{\overline{\mathcal{A}'}}] + L_{\mathcal{B}}[\theta^*_{\overline{\mathcal{A}'}}] \tag{95}$$

Note that we can choose K, K' such that M only depends on $\theta^*_{\overline{\mathcal{A}}\setminus\mathcal{E}} = \theta^*_{\{\mathcal{B},\mathcal{C}\cup\mathcal{D}\}}$ since the ranking over $\theta^*_{\mathcal{B}}$ is separable from the ranking over $\theta^*_{\{\mathcal{C},\mathcal{D}\}}$. More generally, for any subgame $\mathcal{B}$ and its complement $\mathcal{A}\setminus\mathcal{B}$, the transformation $F_{\mathcal{B}}$ only depends on the utility of the subprocess $U[G,\theta^*_{\mathcal{B}}]$ and the measure $\theta^*_{\mathcal{B},\mathcal{A}\setminus\mathcal{B}}$. Therefore, we obtain the representation:

$$K'[F'_{\mathcal{B}}[U[G,\theta^*_{\mathcal{B}}],\theta^*_{\{\mathcal{B},\mathcal{A}\setminus\mathcal{B}\}}] + F'_{\mathcal{C}}[U[G,\theta^*_{\mathcal{C}}],\theta^*_{\{\mathcal{C},\mathcal{A}\setminus\mathcal{C}\}}] + F'_{\mathcal{D}}[U[G,\theta^*_{\mathcal{D}}],\theta^*_{\{\mathcal{D},\mathcal{A}\setminus\mathcal{D}\}}],\theta^*_{\overline{\mathcal{A}'}}] \tag{96}$$

What is left to show is that all $F_{\mathcal{B}}$ and K are affine. Note that in the above representation, we can let $\theta^*_{\{\mathcal{B}\cup\mathcal{C}\cup\mathcal{D}\}}$ converge to $\theta^*_{\{\mathcal{C}\cup\mathcal{D}\}}$ by letting $\mathcal{B}$ become null. On the subset of such processes, we obtain the representation:

$$U[G,\theta^*_{\mathcal{C}\cup\mathcal{D}}] = K'\left[F'_{\mathcal{C}}[U[G,\theta^*_{\mathcal{C}}],\theta^*_{\{\mathcal{C},\mathcal{D}\}}] + F'_{\mathcal{D}}[U[G,\theta^*_{\mathcal{D}}],\theta^*_{\{\mathcal{C},\mathcal{D}\}}],\theta^*_{\{\mathcal{C},\mathcal{D}\}}\right] \tag{97}$$

Noting that Equation (94) is a monotone function of the above, we can substitute and use the uniqueness of additive representation argument to obtain that $F_{\mathcal{C}\cup\mathcal{D}}[K'[r,\theta^*_{\{\mathcal{C},\mathcal{D}\}}],\theta^*_{\{\mathcal{B},\mathcal{C}\cup\mathcal{D}\}}]$ is affine in r. This condition fulfills the functional equation solved in Lemma 2. It follows that there exists some continuous monotone transformation of U that makes both K and all functions $F_{\mathcal{C}}$ affine transformations of their first argument. □

Lemma 10.

$$M_{\mathcal{C}}[\theta^*_{\{\mathcal{B},\mathcal{C},\mathcal{D}\}}] = M_{\mathcal{C}\cup\mathcal{D}}[\theta^*_{\{\mathcal{B},\mathcal{C}\cup\mathcal{D}\}}]M_{\mathcal{C}}[\theta^*_{\{\mathcal{C},\mathcal{D}\}}] \tag{98}$$

$$= M_{\mathcal{C}}[\theta^*_{\{\mathcal{C},\mathcal{B}\cup\mathcal{D}\}}]. \tag{99}$$

Proof. From the previous lemma, we have the following representations:

$$\begin{aligned}
U[G,\theta] =& U[G,\theta^*_{\{\mathcal{B},\mathcal{C},\mathcal{D}\}} \otimes \theta^*_{\mathcal{B}} \otimes \theta^*_{\mathcal{C}} \otimes \theta^*_{\mathcal{D}}] \\
=& U[G,\theta^*_{\mathcal{C}}]M_{\mathcal{C}}[\theta^*_{\{\mathcal{B},\mathcal{C},\mathcal{D}\}}] + \dots \\
=& U[G,\theta^*_{\{\mathcal{B},\mathcal{C}\cup\mathcal{D}\}} \otimes \theta^*_{\mathcal{B}} \otimes \theta^*_{\{\mathcal{C},\mathcal{D}\}} \otimes \theta^*_{\mathcal{C}} \otimes \theta^*_{\mathcal{D}}] \\
=& U[G,\theta^*_{\mathcal{C}\cup\mathcal{D}}]M_{\mathcal{C}\cup\mathcal{D}}[\theta^*_{\{\mathcal{B},\mathcal{C}\cup\mathcal{D}\}}] + \dots \\
=& U[G,\theta^*_{\mathcal{C}}]M_{\mathcal{C}}[\theta^*_{\{\mathcal{C},\mathcal{D}\}}]M_{\mathcal{C}\cup\mathcal{D}}[\theta^*_{\{\mathcal{B},\mathcal{C}\cup\mathcal{D}\}}] + \dots, \qquad (100)
\end{aligned}$$

where we have assumed without loss of generality that the game form G is sufficiently rich in actions. Using a small change in $U[G,\theta^*_{\mathcal{C}}]$, the first line of the result follows since the change must be equal in all of the above representations. The second line result can be derived by comparing the above representation to:

$$U[G,\theta] = U[G,\theta^*_{\{\mathcal{C},\mathcal{B}\cup\mathcal{D}\}} \otimes \theta^*_{\mathcal{C}} \otimes \theta^*_{\{\mathcal{B},\mathcal{D}\}} \otimes \theta^*_{\mathcal{B}} \otimes \theta^*_{\mathcal{D}}] \qquad (101)$$

$$= M_{\mathcal{C}}[\theta^*_{\{\mathcal{C},\mathcal{B}\cup\mathcal{D}\}}]U[G,\theta^*_{\mathcal{C}}] + \dots \qquad (102)$$

By a small change in $U[G,\theta^*_{\{\mathcal{C},\mathcal{B}\cup\mathcal{D}\}}]$, it then follows that $M_{\mathcal{C}}[\theta^*_{\{\mathcal{B},\mathcal{C},\mathcal{C}\}}] = M_{\mathcal{C}}[\theta^*_{\{\mathcal{C},\mathcal{B}\cup\mathcal{D}\}}]$. □

Lemma 11. $M_{\mathcal{C}}[\theta^*_{\{\mathcal{C},\mathcal{D}\}}] = \sum_\mu \theta^*_{\{\mathcal{C},\mathcal{D}\}}[\mu]\mu[\mathcal{C}]$.

Proof. We use the special case of a process with a support of three subprocesses and in which only a single strategy μ' yields with positive probability the subprocess obtained by conditioning on the subgame $\mathcal{C}$. Formally, let θ fulfill for all μ:

$$\theta[\mu] > 0, \mu[\mathcal{C}] > 0 \Rightarrow \mu = \mu' \qquad (103)$$

from this follows that we can parametrize the following measures:

$$\theta^*_{\{\mathcal{C},\mathcal{B}\cup\mathcal{D}\}} = f[\theta[\mu'], \mu'[\mathcal{C}]] \qquad (104)$$

$$\theta^*_{\{\mathcal{C},\mathcal{B}\}} = g\left[\frac{\theta[\mu'](\mu'[\mathcal{B}\cup\mathcal{C}])}{\sum_\mu \theta[\mu](\mu[\mathcal{B}\cup\mathcal{C}])}, \frac{\mu'[\mathcal{C}]}{1-\mu'[\mathcal{B}\cup\mathcal{C}]}\right] \qquad (105)$$

$$\theta^*_{\{\mathcal{B}\cup\mathcal{C},\mathcal{D}\}} = h\left[\{\theta[\mu], \mu[\mathcal{B}\cup\mathcal{C}]\}_{\mu\in supp[\theta]}\right] \qquad (106)$$

We therefore have:

$$
\begin{aligned}
&M_{\mathcal{C}}[\theta^*_{\{\mathcal{C},\mathcal{B}\cup\mathcal{D}\}}] && (107)\\
=&M_{\mathcal{C}}[f[\theta[\mu'],\mu'[\mathcal{C}]]] && (108)\\
=&M_{\mathcal{C}}[\theta^*_{\{\mathcal{B}\cup\mathcal{C}\}}]M_{\mathcal{B}\cup\mathcal{C}}[\theta^*_{\{\mathcal{B}\cup\mathcal{C},\mathcal{D}\}}] && (109)\\
=&M_{\mathcal{C}}\left[g\left[\frac{\theta[\mu'](\mu'[\mathcal{B}\cup\mathcal{C}])}{\sum_\mu \theta[\mu](\mu[\mathcal{B}\cup\mathcal{C}])},\frac{\mu'[\mathcal{C}]}{\mu'[\mathcal{B}\cup\mathcal{C}]}\right]\right]M_{\mathcal{B}\cup\mathcal{C}}\left[h\left[\{\theta[\mu],\mu[\mathcal{B}\cup\mathcal{C}]\}_{\mu\in supp[\theta]}\right]\right] && (110)
\end{aligned}
$$

Note that only $M_{\{\mathcal{C},\mathcal{D}\}}$ and $M_{\{\mathcal{B},\mathcal{C},\mathcal{D}\}}$ depend on $\mu'[\mathcal{C}]$. We apply the following substitutions:

$$
\begin{aligned}
\sum_\mu \theta[\mu]\mu[\mathcal{B}\cup\mathcal{C}] =&\hat{p} && (111)\\
\frac{\theta[\mu']\mu'[\mathcal{B}\cup\mathcal{C}]}{\hat{p}} =&\hat{\theta} && (112)
\end{aligned}
$$

and obtain:

$$
\begin{aligned}
&M_{\mathcal{C}}[f[\theta[\mu'],\mu'[\mathcal{C}]]] && (113)\\
=&M_{\mathcal{C}}\left[g\left[\hat{\theta},\frac{\theta[\mu']\mu'[\mathcal{C}]}{\hat{\theta}\hat{p}}\right]\right]M_{\mathcal{B}\cup\mathcal{C}}\left[h\left[\{\theta[\mu],\mu[\mathcal{B}\cup\mathcal{C}]\}_{\mu\in supp[\theta]}\right]\right] && (114)
\end{aligned}
$$

It follows that the composition of $M_{\mathcal{B}\cup\mathcal{C}}$ and h only depends on the values of $\theta[\mu']$, $\mu'[\mathcal{B}\cup\mathcal{C}]$, $\hat{p}$, $\hat{\theta}$, which we write as

$$
M_{\mathcal{B}\cup\mathcal{C}}\left[h\left[\{\theta[\mu],\mu[\mathcal{B}\cup\mathcal{C}]\}_{\mu\in supp[\theta]}\right]\right] = k[\theta[\mu'],\mu'[\mathcal{B}\cup\mathcal{C}],\hat{p},\hat{\theta}]. \quad (115)
$$

Holding $\theta[\mu']$ and $\hat{\theta}$ constant we obtain a Pexider-like logarithmic equation[21] with the solution,

$$
\begin{aligned}
M_{\mathcal{C}}[f[\theta[\mu'],\mu'[\mathcal{C}]]] =&\hat{f}[\theta[\mu']]\mu'[\mathcal{C}]^\gamma && (116)\\
M_{\mathcal{C}}\left[g\left[\hat{\theta},\frac{\theta[\mu']\mu'[\mathcal{C}]}{\hat{\theta}\hat{p}}\right]\right] =&\hat{g}[\hat{\theta}]\left(\frac{\theta[\mu']\mu'[\mathcal{C}]}{\hat{\theta}\hat{p}}\right)^\gamma && (117)\\
k[\theta[\mu'],\mu'[\mathcal{C}],\hat{p},\hat{\theta}] =&\hat{k}[\theta[\mu'],\hat{\theta}']\,(\hat{p})^\gamma && (118)
\end{aligned}
$$

Next, we note that $\hat{k}[\cdot] = 1$ since in the limit if $\hat{p} \to 1$, the functions

[21] We obtain exactly the Pexider equation on an interval domain by taking logarithms on both sides, rearranging terms, and exponentiating the variables. Solving this functional equation yields the stated result.

$M_{\mathcal{C}}[f[\ldots]]$ and $M_{\mathcal{C}}[g[\ldots]]$ converge. It follows that $\hat{f}[\theta[\mu']] = (\theta[\mu'])^{\gamma}$ and $\hat{g}[\hat{\theta}] = (\hat{\theta})^{\gamma}$.

Since we have previously obtained in the proof of Lemma 3 an expected utility representation over outcomes, and outcomes are subgames, it follows directly that $\gamma = 1$. □

A.5 Joint Conditional Additivity on Outcomes and Strategies

We now have two representations, one conditionally additively separable in strategies and the other additively separable across outcomes. The two representations are affine transformations of another as shown in the proof of 4. Without loss of generality, we assume this transformation to be the identity transformation. We then obtain the following lemma.

Lemma 12. *If $\overline{\mathcal{A}}$ is a partition of $\mathcal{A}$ into disjoint subgames, then*

$$U_i[G,\theta] = \sum_{\mu_i} \sum_{\mathcal{B}\in\overline{\mathcal{A}}} \theta_i[\mu_i] \left(\mu_i[\mathcal{B}]U[(G,\theta)|\mathcal{B}] + l[\mu_i[\mathcal{B}],\rho[\mathcal{B}]]\right) \tag{119}$$

Proof. We have two representations from previous Lemmas:

$$\sum_{\mu_i} \theta[\mu_i] v_i[\rho|\mu_i,\rho] = \sum_{\mathcal{B}} \rho(\mathcal{B}) U[(G,\theta)|\mathcal{B}] + L_{\mathcal{B}}[\theta] \tag{120}$$

We assume there are four subprocesses, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$. Without loss of generality, we assume that these subprocesses each yield a different outcome with certainty and thus $\rho[x|\mu] = \mu[\mathcal{B}]$ for some outcome x. We can choose a parametrization such that $\rho|\mu_i = f(\epsilon,\delta)$, $\rho|\mu_i' = f'(\epsilon)$, and $\rho|\mu_i'' = f''(\delta)$. Namely, we choose ϵ to transfer probability from $\mu_i[\mathcal{B}]$ to $\mu_i[\mathcal{C}]$ and from $\mu_i'[\mathcal{C}]$ to $\mu_i'[\mathcal{B}]$ to keep the probabilities of the outcomes unchanged. δ reallocates probability from $\mu_i[\mathcal{D}]$ to $\mu_i[\mathcal{E}]$ and from $\mu_i''[\mathcal{E}]$ to $\mu_i''[\mathcal{D}]$. Moreover, $\theta|\mathcal{B} = t(\epsilon)$ and $\theta|\mathcal{B} = t'(\epsilon)$ as well as $\theta|\mathcal{C} = t''(\delta)$ and $\theta|\mathcal{D} = t'''(\delta)$.

$$\theta[\mu_i] v_i[f(\epsilon,\delta),\rho] \tag{121}$$
$$+ \theta[\mu_i'] v_i'[f'(\epsilon),\rho] \tag{122}$$
$$+ \theta[\mu_i''] v_i''[f''(\delta),\rho] + \ldots \tag{123}$$
$$= L_{\mathcal{B}}[t(\epsilon)] + L_{\mathcal{C}}[t'(\epsilon)] + L_{\mathcal{D}}[t''(\delta)] + L_{\mathcal{E}}[t'''(\delta)] + \ldots \tag{124}$$

and therefore v_i is additively separable in ϵ and δ. Repeating the above steps for reassignments of probability between $\mu_i[\mathcal{B}]$ and $\mu_i[\mathcal{D}]$ as well as between $\mu_i[\mathcal{C}]$ and $\mu_i[\mathcal{E}]$, it is straightforward to obtain that v_i is indeed additively separable across $\mu_i[\mathcal{B}]$, $\mu_i[\mathcal{C}]$, etc.. Thus, $v_i[\rho|\mu_i,\rho] = \sum_{\mathcal{B}} w_{i,\mathcal{B}}[\rho|\mu_i[\mathcal{B}],\rho]$. We can now derive the functional form of $L_{\mathcal{B}}$.

$$
\begin{aligned}
& L_{\mathcal{B}}[\theta] + L_{\mathcal{C}}[\theta] + \dots && (125)\\
=& \sum_{\mu} \theta[\mu] \left(w_{i,\mathcal{B}}[\mu[\mathcal{B}],\rho] - \mu[\mathcal{B}] U_i[(G,\theta)|\mathcal{B}] \right) && (126)\\
& + \sum_{\mu} \theta[\mu] \left(w_{i,\mathcal{C}}[\mu[\mathcal{C}],\rho] - \mu[\mathcal{B}] U_i[(G,\theta)|\mathcal{C}] \right) + \dots && (127)
\end{aligned}
$$

For fixed $\rho[\mathcal{B} \cup \mathcal{C}]$ and considering only changes in $\theta_{\mathcal{B},\mathcal{C}}$, $L_{\mathcal{B}}$ does not depend on any of the omitted terms. Moreover, $L_{\mathcal{B}}$ does not depend on $\rho[C]$ since $L_{\mathcal{B}}[\theta] = L_{\mathcal{B}}[\theta_{\mathcal{B},\mathcal{C}\cup\mathcal{D}\cup\mathcal{E}}]$. Therefore, for a suitably chosen function $l_{\mathcal{B}}$, we have that $L_{\mathcal{B}}[\theta] = \sum_{\mu} \theta[\mu] l_{\mathcal{B}}[\mu[\mathcal{B}],\rho[\mathcal{B}]]$. By Continuity, we may impose without loss of generality that $l_{\mathcal{B}}[0,\rho[\mathcal{B}]] = 0$.

What is left to show is that $l_{\mathcal{B}}$ can be chosen to be identical across $\mathcal{B}$. For this, suppose that the support of the subprocess $\mathcal{B}$ contains two outcomes. By Continuity, for a sequence of subprocesses such that the probability of one outcome converges to zero, their utility $U_i[(G,\theta)|\mathcal{B}]$ converges to the utility at which the probability of the outcome is zero. Similarly, $U_i[G,\theta]$ converges to the utility at which the probability of the outcome is zero. But then under an outcome equivalent transformation $f_{\mathcal{B}}[\dots] = f_{\mathcal{B}'}[\dots]$ where $\mathcal{B}'$ is obtained by removing actions that yield the outcome. □

A.6 Determination of the Functional Form of Procedural Preferences

We now define $h[x,y] = l[x,y] + l[1-x,1-y]$.

Lemma 13.

$$h(1,x) = \alpha(x \ln[x] + (1-x)\ln[1-x]) \tag{128}$$

Proof. We employ a process such that $\theta[\mu] + \theta[\mu'] + \theta[\mu''] = 1$. Also, $\mu[\mathcal{B}] = \mu'[\mathcal{C}] = \mu''[\mathcal{D}] = 1$. We use the above lemma twice on different partitions, $\{\mathcal{B}, \mathcal{C} \cup \mathcal{D}\}$ and $\{\mathcal{C}, \mathcal{B} \cup \mathcal{D}\}$. We therefore obtain the two representations:

$$\theta[\mu]h(1,\theta[\mu]) + (1-\theta[\mu])h(0,\theta[\mu]) \tag{129}$$

$$+ (1-\theta[\mu])\left(\frac{\theta[\mu']}{1-\theta[\mu]}h(1,\frac{\theta[\mu']}{1-\theta[\mu]}) + \frac{\theta[\mu'']}{1-\theta[\mu]}h(0,\frac{\theta[\mu'']}{1-\theta[\mu]})\right) \tag{130}$$

$$=\theta[\mu']h(1,\theta[\mu']) + (1-\theta[\mu'])h(0,\theta[\mu']) \tag{131}$$

$$+ (1-\theta[\mu'])\left(\frac{\theta[\mu]}{1-\theta[\mu']}h(1,\frac{\theta[\mu]}{1-\theta[\mu']}) + \frac{\theta[\mu'']}{1-\theta[\mu']}h(0,\frac{\theta[\mu'']}{1-\theta[\mu']})\right) \tag{132}$$

Where we have cancelled the terms containing $U[G,\theta_{\mathcal{B}}]$, etc.. We may assume without loss of generality that $h(1,x) = h(0,x)$. We then have:

$$h(1,\theta[\mu]) + (1-\theta[\mu])\left(h(1,\frac{\theta[\mu']}{1-\theta[\mu]})\right) \tag{133}$$

$$=h(1,\theta[\mu']) + (1-\theta[\mu'])\left(h(1,\frac{\theta[\mu]}{1-\theta[\mu']})\right) \tag{134}$$

This is the fundamental equation of information (Aczél & Dhombres, 1989; Ebanks, Kannappan, & Ng, 1987). Up to a constant and a linear component in $\theta[\mu]$ (which can be removed by redefining $U[G,\theta_{\mathcal{B}}]$), the solution is:

$$h(1,\theta[\mu]) = \alpha\left(\theta[\mu]\ln[\theta[\mu]] + (1-\theta[\mu])\ln[1-\theta[\mu]]\right) \tag{135}$$

□

Lemma 14.

$$h(x,x) = \beta\left(x\ln[x] + (1-x)\ln[1-x]\right) \tag{136}$$

Proof. We employ a process with $\theta[\mu] = 1$, $\mu[\mathcal{B}] + \mu[\mathcal{C}] + \mu[\mathcal{D}] = 1$. The two partitions $\{\mathcal{B}, \mathcal{C}\cup\mathcal{D}\}$ and $\{\mathcal{C}, \mathcal{B}\cup\mathcal{D}\}$ generate two representations which yield after cancelling terms:

$$h(\mu[\mathcal{B}],\mu[\mathcal{B}]) + (1-\mu[\mathcal{B}])\left(h(\frac{\mu[\mathcal{C}]}{1-\mu[\mathcal{B}]},\frac{\mu[\mathcal{C}]}{1-\mu[\mathcal{B}]})\right) \tag{137}$$

$$=h(\mu[\mathcal{C}],\mu[\mathcal{C}]) + (1-\mu[\mathcal{C}])\left(h(\frac{\mu[\mathcal{B}]}{1-\mu[\mathcal{C}]},\frac{\mu[\mathcal{B}]}{1-\mu[\mathcal{C}]})\right) \tag{138}$$

This is the fundamental equation of information with the solution:

$$h(\mu[\mathcal{B}],\mu[\mathcal{B}]) = \alpha\left(\mu[\mathcal{B}]\ln[\mu[\mathcal{B}]] + (1-\mu[\mathcal{B}])\ln[1-\mu[\mathcal{B}]]\right) \tag{139}$$

$\square$

Lemma 15.

$$h[x,y] =(\beta-\alpha)\left(x\ln[x]+(1-x)\ln[1-x]\right) \tag{140}$$
$$+\alpha\left(y\ln[y]+(1-y)\ln[1-y]\right) \tag{141}$$

Proof. We employ a process of the form $\theta[\mu]+\theta[\mu']=1$ with strategies that fulfill: $\mu[\mathcal{B}\cup\mathcal{D}]=1$ and $\mu'[\mathcal{C}]=1$. We obtain the representations:

$$\theta[\mu]h[\mu[\mathcal{B}],\theta[\mu]\mu[\mathcal{B}]]+(1-\theta[\mu])h[0,\theta[\mu]\mu[\mathcal{B}]]+(1-\theta[\mu]\mu[\mathcal{B}]) \tag{142}$$
$$\cdot\left(\frac{\theta[\mu](1-\mu[\mathcal{B}])}{1-\theta[\mu]\mu[\mathcal{B}]}h[0,\frac{1-\theta[\mu]}{1-\theta[\mu]\mu[\mathcal{B}]}]+\left(\frac{1-\theta[\mu]}{1-\theta[\mu]\mu[\mathcal{B}]}\right)h[1,\frac{1-\theta[\mu]}{1-\theta[\mu]\mu[\mathcal{B}]}]\right) \tag{143}$$
$$=\theta[\mu]h[0,1-\theta[\mu]]+(1-\theta[\mu])h[1,1-\theta[\mu]]+\theta[\mu]h[\mu[\mathcal{B}],\mu[\mathcal{B}]] \tag{144}$$

For better readability, we substitute $x=\mu[\mathcal{B}]$ and $y=\theta[\mu]\mu[\mathcal{B}]$.

$$y/xh[x,y]+(1-y/x)h[0,y]+(1-y)h[1,\frac{1-x}{1-y}\frac{y}{x}] \tag{145}$$
$$=h[0,1-y/x]+(y/x)h[x,x] \tag{146}$$

where we made use of $h[0,x]=h[1,x]$. We next solve for $h[x,y]$:

$$h[x,y] =x/yh[0,1-y/x] \tag{147}$$
$$+h[x,x] \tag{148}$$
$$-(x-y)/yh[0,y] \tag{149}$$
$$-x(1-y)/yh[1,\frac{1-x}{1-y}\frac{y}{x}] \tag{150}$$

From Lemmas 13 and 14 we have the solutions for $h[0,y]$, and $h[x,x]$. Substituting these into the above equation gives us the solution for $h[x,y]$:

$$h[x,y] =(\beta-\alpha)\left(x\ln[x]+(1-x)\ln[1-x]\right) \tag{151}$$
$$+\alpha\left(y\ln[y]+(1-y)\ln[1-y]\right) \tag{152}$$
$$=h[x,x]-h[0,x]+h[0,y] \tag{153}$$

It is straightforward to verify that the solutions for $h[x,x]$, $h[0,y]$, and $h[x,y]$ are compatible with another. $\square$

We conclude the proof as follows. If $\beta = 0$, then the procedural preferences are equal to the binary mutual information. We obtain that $\beta = 0$ by lottery independence. If $\beta \neq \alpha$, then $U[D_{\mathcal{N}}(G, \theta)]$ consists of an expectation and an entropy, violating Lottery Independence. Extending the binary mutual information to multiple outcomes follows from substituting the utility representation of the subprocesses. We have therefore identified that U_i is the sum of expectations across outcomes and mutual information. Since we determine the function h for each player, we may choose separate parameters β and name these d_i.

B Proof of Propositions

Proof. The consumer i maximizes the Lagrangian:

$$\max_{x_i, y_i, \lambda_i} u(x_i, y_i) - \lambda_i (x_i p - (w_i y_i)^{1-r} \frac{f(r, \beta, \delta)}{1-r}) \tag{154}$$

As long as $r > -\frac{\zeta+\eta}{1-\eta}$, which is guaranteed by $0 < \eta < 1$, $\zeta > 0$, and $0 < r < 1$, this problem has an interior solution and first order conditions are necessary and sufficient for optimality:

$$w_i^{1-r} f(r, \beta, \delta) \lambda_i = y_i^{\zeta+r} \tag{155}$$

$$\lambda_i p = \alpha_i \delta^{1-\eta} x_i^{-\eta} \tag{156}$$

$$x_i p = (w_i y_i)^{1-r} \frac{f(r, \beta, \delta)}{1-r} \tag{157}$$

From which we obtain:

$$\frac{w_i^{1-r} f(r, \beta, \delta)}{p} = \frac{y_i^{\zeta+r}}{\alpha_i \delta^{1-\eta} x_i^{-\eta}} \tag{158}$$

$$\alpha_i \delta^{1-\eta} x_i^{1-\eta} = \frac{y_i^{1+\zeta}}{1-r} \tag{159}$$

Solving for the optimal consumption and labor plan:

$$x_i^* = \left(\frac{(\alpha_i \delta^{1-\eta})^{1-r} \left(\frac{f(r,\beta,\delta) w_i^{1-r}}{p} \right)^{\zeta+1}}{(1-r)^{\zeta+r}} \right)^{\frac{1}{\zeta+\eta(1-r)+r}} \tag{160}$$

$$y_i^* = \left(\frac{\alpha_i \delta^{1-\eta} \left(\frac{f(r,\beta,\delta) w_i^{1-r}}{p} \right)^{1-\eta}}{(1-r)^{-\eta}} \right)^{\frac{1}{\zeta+\eta(1-r)+r}} \tag{161}$$

The firm takes prices as given and maximizes the objective:

$$\max_{C,\{y_i\}_{i\in[0,1]},\lambda} Cp - \int_0^1 w_i y_i di - \lambda \left(C - \int_0^1 y_i \beta \gamma_i di \right) \tag{162}$$

from which we obtain the optimality condition:

$$w_i = p\beta\gamma_i \tag{163}$$

In equilibrium with numeraire $p = 1/\beta$ then:

$$x_i^* = \left(\frac{(\alpha_i \delta^{1-\eta})^{1-r} \left(\beta f(r,\beta,\delta) \gamma_i^{1-r} \right)^{\zeta+1}}{(1-r)^{\zeta+r}} \right)^{\frac{1}{\zeta+\eta(1-r)+r}} \tag{164}$$

$$y_i^* = \left(\frac{\alpha_i \delta^{1-\eta} \left(\beta f(r,\beta,\delta) \gamma_i^{1-r} \right)^{1-\eta}}{(1-r)^{-\eta}} \right)^{\frac{1}{\zeta+\eta(1-r)+r}} \tag{165}$$

We assume a market clearing condition according to which the government chooses $f(r,\beta,\delta)$ such that it purchases a fraction $\bar{g}$ of the good. We therefore have the market clearing and budget balance constraints:

$$\int x_i di = (1-\bar{g}) \int \beta \gamma_i y_i di \tag{166}$$

$$p\bar{g}q = \int w_i y_i di - \int (w_i y_i)^{1-r} \frac{f(r,\beta,\delta)}{1-r} di \tag{167}$$

From which we obtain:

$$\frac{f(r,\beta,\delta)}{1-r} = (1-\bar{g})\frac{\int y_i\gamma_i di}{\int (\gamma_i y_i)^{1-r} di} \tag{168}$$

$$\frac{f(r,\beta,\delta)^{\frac{\zeta+\eta}{\zeta+\eta(1-r)+r}}}{1-r} = (1-\bar{g})\left(\frac{(\beta\delta)^{1-\eta}}{(1-r)^{-\eta}}\right)^{\frac{r}{\zeta+\eta(1-r)+r}} \frac{\int \left(\alpha_i\gamma_i^{1+\zeta}\right)^{\frac{1}{\zeta+\eta(1-r)+r}} di}{\int \left(\alpha_i\gamma_i^{1+\zeta}\right)^{\frac{1-r}{\zeta+\eta(1-r)+r}} di} \tag{169}$$

As levels can be shown to be irrelevant for the mutual information of lognormal variables, we are only interested in the proportionality:

$$f(r,\beta,\delta) \propto (\beta\delta)^{r\frac{1-\eta}{\zeta+\eta}} \tag{170}$$

and therefore

$$x_i^* \propto \left(\frac{\alpha_i^{1-r}\delta^{1-\eta+r\frac{(1-\eta)^2}{\zeta+\eta}}\left(\beta^{1+r\frac{1-\eta}{\zeta+\eta}}\gamma_i^{1-r}\right)^{\zeta+1}}{(1-r)^{\zeta+r}}\right)^{\frac{1}{\zeta+\eta(1-r)+r}} \tag{171}$$

$$y_i^* \propto \left(\frac{\alpha_i\left((\beta\delta)^{1+r\frac{1-\eta}{\zeta+\eta}}\gamma_i^{1-r}\right)^{1-\eta}}{(1-r)^{-\eta}}\right)^{\frac{1}{\zeta+\eta(1-r)+r}} \tag{172}$$

Taking logarithms and assuming proportionality factors $\hat{Q}$ and $\bar{Q}$:

$$\begin{aligned}\ln x_i^* = \hat{Q} &+ \frac{1-r}{\zeta+\eta(1-r)+r}\ln\alpha_i \\ &+ \frac{\zeta+1}{\zeta+\eta}\ln\beta \\ &+ \frac{(1-r)(\zeta+1)}{\zeta+\eta(1-r)+r}\ln\gamma_i \\ &+ \frac{1-\eta}{\zeta+\eta}\ln\delta\end{aligned} \tag{173}$$

$$
\begin{aligned}
\ln y_i^* = \bar{Q} &+ \frac{1}{\zeta + \eta(1-r) + r} \ln \alpha_i \\
&+ \frac{1-\eta}{\zeta + \eta} \ln \beta \\
&+ \frac{(1-r)(1-\eta)}{\zeta + \eta(1-r) + r} \ln \gamma_i \\
&+ \frac{1-\eta}{\zeta + \eta} \ln \delta
\end{aligned}
\tag{174}
$$

$\ln \alpha_i$, $\ln x_i^*$, $\ln y_i^*$ are jointly normal with covariance matrix:

$$
\begin{pmatrix}
\sigma_\alpha^2 & & \\
\frac{(1-r)\sigma_\alpha^2}{\zeta+\eta(1-r)+r} & \frac{(1-r)^2(\sigma_\alpha^2+(1+\zeta)^2\sigma_\gamma^2)}{(\zeta+\eta(1-r)+r)^2} + \frac{(1+\zeta)^2\sigma_\beta^2+(1-\eta)^2\sigma_\delta^2}{(\zeta+\eta)^2} & \\
\frac{\sigma_\alpha^2}{\zeta+\eta(1-r)+r} & \frac{(1-r)(\sigma_\alpha^2+(1-r)(1-\eta)(1+\zeta)\sigma_\gamma^2)}{(\zeta+\eta(1-r)+r)^2} + \frac{(1-\eta)(1+\zeta)\sigma_\beta^2+(1-\eta)^2\sigma_\delta^2}{(\zeta+\eta)^2} & \frac{\sigma_\alpha^2+(1-r)^2(1-\eta)^2\sigma_\gamma^2}{(\zeta+\eta(1-r)+r)^2} + \frac{(1-\eta)^2\left(\sigma_\beta^2+\sigma_\delta^2\right)}{(\zeta+\eta)^2}
\end{pmatrix}
\tag{175}
$$

To determine the amount of consumption and labor freedom, we employ two important facts. First, mutual information is invariate under homeomorphisms (Kraskov, Stögbauer, & Grassberger, 2004) and therefore the mutual information of the logarithms of the variables is identical to the mutual information between the variables themselves. Second, for jointly normal variables, the mutual information between two variables x, y can be conveniently calculated by $MI(x,y) = -\frac{1}{2}\ln[1 - corr(x,y)^2]$. Calculating the correlation between α_i and x_i^* and between α_i and y_i^* therefore yields the Propositions 1 and 2. Proposition 3 follows directly from the first order conditions of maximizing labor freedom with respect to r.

To derive Proposition 4, we employ the conditioning property of mutual information: $MI(x,(y,z)) = MI(x,y) + MI(x,z|y)$ where $MI(x,z|y)$ is the expectation (with respect to y) of the mutual information between x and z using the conditional distribution of x and z given y. The closed form expression for the conditional mutual information between α_i and x_i^* given y_i^* would not fit this page but can be easily calculated using symbolic mathematics software. After cancelling terms, we obtain the value of the freedom measure:

$$
-(1/2)\ln[\frac{\sigma_\delta^2(-1+\eta)^2}{\sigma_\alpha^2 + \sigma_\delta^2(-1+\eta)^2}]. \tag{176}
$$

□