Contests with Group-Specific Public Goods and Complementarities in Efforts

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Abstract

This paper starts from the observation that in public-goods group contests, group impact can in general not be additively decomposed into some sum (of functions) of individual efforts. We use a CES-impact function to identify the main channels of influence of the elasticity of substitution on the behavior in and the outcome of such a contest. We characterize the Nash equilibria of this game and carry out comparative-static exercises with respect to the elasticity of substitution among group members' efforts. If groups are homogeneous (i.e. all group members have the same valuation and efficiency within the group), the elasticity of substitution has no effect on the equilibrium. For heterogeneous groups, the higher the complementarity of efforts of that group, the lower the divergence of efforts among group members and the lower the winning probability of that group. This contradicts the common intuition that groups can improve their performance by solving the free-rider problem via higher degrees of complementarity of efforts.

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1 Introduction

In many economic situations like R&D races, military conflicts, lobbying, or sports, groups compete for economic rents that are group-specific public goods. Usually, in all these examples, efforts of different group members are to some extent complementary. In R&D races, where teams of researchers develop new technologies, the whole project is often divided into different, more or less complementary sub-projects that are carried out by different researchers. In military conflicts the armed forces are highly specialized and often divided into complementary units. The same is true for the standard lobbying case if representatives of different firms or organizations lobbying for the same policy differ in qualifications and specialize accordingly. In sports contests, team members are usually specialized with respect to qualifications that complement each other in a non-additive way. Another example for a group conflict is competition for a prize between different business partnerships. Management consultants, lawyers, physicians, and architects often organize their companies as partnerships where individual incomes of the partners are determined according to their shares in the partnership (Garicano and Santos (2004)). Consultancies and architect offices competing for projects and physicians competing for patients are all in situations that closely resemble a contest.[1] Furthermore, the substitutability of the partners' efforts depends on the industry as well as on the qualifications of the different partners (and thereby the organizational structure and the business strategy). A medical center that combines physicians with different qualifications has a relatively high degree of complementarity between the different physicians' qualifications. A consulting firm that specializes in only one field of business, on the other hand, is likely to have a higher degree of substitutability between the partners' efforts.

This list of examples could be more or less arbitrarily extended because the mere idea of specialization implies that there is a certain degree of complementarity in team or group production. Individuals differ in talents, qualifications, and affections in a way that they will specialize to increase overall productivity. We can therefore expect a certain degree of complementarity between the efforts of the group members. Alchian and Demsetz (1972) see the non-additivity as constitutive for group or

[1] Competition for customers has more the character of an oligopolistic market. However, if market demand is isoelastic, the Tullock contest is isomorphic to a Cournot oligopoly.

team production (pp. 777): *"Resource owners increase productivity through cooperative specialization. [...] With team production it is difficult, solely by observing total output, to either define or determine each individual's contribution to this output of the cooperating inputs. The output is yielded by a team, by definition, and it is not a sum of separable outputs of each of its members. [...] Usual explanations of the gains from cooperative behavior rely on exchange and production in accord with the comparative advantage specialization principle with separable additive production. However [...] there is a source of gain from cooperative activity involving working as a team, wherein individual cooperating inputs do not yield identifiable, separate products which can be summed to measure the total output."*

Despite the growing interest in the influence of heterogeneity within and between groups, with only a few exceptions the literature on group contests (surveyed in Corchón (2007); Garfinkel and Skaperdas (2007); Konrad (2009)) has focused attention on situations where the effort levels of group members are perfect substitutes. This case is an important starting point for the analysis of group contests. However, if complementarities are the rule rather than the exception, it is important to understand how the degree of complementarity between individual efforts influences behavior in and the outcome of the contest.

We use a CES production (impact) function in an n-group contest. To be more specific, assume that individual efforts x_i^k are mapped onto group impact (that enters a lottery contest as aggregate contribution by a group)[2] by means of a CES-impact function, $g_i \cdot (\sum a_i^k \cdot (x_i^k)^{\gamma_i})^{1/\gamma_i}$, with variable elasticity of substitution $1/(1-\gamma_i)$, ranging from perfect complements ($\gamma_i \to -\infty$) to perfect substitutes ($\gamma_i \to 1$), and aggregate as well as individual efficiency parameters g_i, a_i^k respectively. The contest is of the Tullock type, and the rent is a group-specific public good (i.e. nonrival and nonexcludable in consumption).

If groups instead of individuals compete in a contest, the well-known free-rider problem among group members exists. Every individual bears the full costs of its investments, whereas the benefits partly spill over to the rest of the group (Katz et al. (1990); Esteban and Ray (2001); Epstein and Mealem (2009); Nitzan and Ueda (2009); Ryvkin (2011)). Depending on the sharing rule applied, this problem may also exist for a private good (Nitzan (1991a,b); Esteban and Ray (2001); Nitzan and Ueda (2009)). In the recent literature, Baik (2008), Epstein and Mealem (2009), and

[2]The term 'impact function' is defined and discussed in Wärneryd (2001); Münster (2009).

Lee (2012) have presented contest models with group-specific public goods. A major result in Baik (2008) is that in a model with linear effort costs and additively linear impact functions only those group members with the highest valuation of the rent make positive investments in the contest. In his model, efforts of group members are perfect substitutes and therefore the optimality conditions given by the first-order conditions cannot hold for different valuations. With several group members having the maximal valuation among the group, there exist multiple equilibria, since the first order condition only defines the total effort spent by the group. Epstein and Mealem (2009) stick to the assumption of additive separability of individual effort in the group-production functions but introduce decreasing returns to investment. Using a technology that fulfills standard "Inada" conditions they show that every individual makes positive investments. Their model is isomorphic to a model with linear impact functions and in which individuals face strictly convex costs. In this sense, effort levels are no longer perfect substitutes, but the impact function is still additively separable. Lee (2012) focuses attention on weakest-link or perfectly complementary impact functions. The perfect complementarity of efforts creates a coordination problem between group members which gives rise to multiple equilibria, and the equilibrium with highest efforts is determined by the valuation of the player with minimum valuation within each group. Hence, the models of Baik (2008) and Lee (2012) represent the "polar" cases with respect to the elasticity of substitution between group members for those cases where the iso-impact curves remain convex. Chowdhury et al. (2011) nicely complements our paper. They analyze the case of a best-shot impact function as the most extreme case of non-convex iso-impact curves.

Our model generalizes the "convex" models by allowing for degrees of complementarity among group efforts. It turns out that the equilibrium behavior of each group is unique for all values of $\gamma_i \in (0, 1)$. For $\gamma_i \in (-\infty, 0)$, the complementarity of efforts is high enough, such that the effort contributions of each member become indispensable. Groups may therefore end up in a high effort equilibrium, in which all members contribute, or in a low effort equilibrium, where none contribute. However, in both cases we can give analytical expressions for equilibrium strategies. In our comparative statics analysis we therefore track equilibria with the same set of groups which fail to coordinate on a high effort equilibrium.

A first corollary is that if there is no within-group heterogeneity with respect to valuations of the prize v_i^k and efficiency a_i^k of each group member and all groups have the same size, the equilibrium is independent of the elasticity of substitution except for the mentioned multiple equilibria issue. This result is a useful starting point because it shows that the elasticity of substitution *per se* has no impact on behavior in the contest, contrary to the cursory idea that increasing the degree of complementarity between group-members' efforts may help to internalize the existing free-rider problem.[3] This point, which has been derived for public-goods games with effort complementarity (Cornes (1993); Cornes and Hartley (2007)), carries over to the contest environment.[4] As a convenient side effect, this independence shows that the standard results on group contests are robust with respect to variations in the elasticity of substitution under within-group homogeneity.

The comparative-static analysis of the paper reveals that this effect is even more pronounced in the general case: A larger degree of complementarity within a single group reduces its winning probability. The intuition for this result is as follows. It is true that a larger degree of complementarity brings the effort levels of the group members closer together. Free-riding that is especially pronounced in the boundary case $\gamma_i = 1$ is therefore mitigated. However, the level of effort is increasingly determined by the group member with the lowest valuation, and it is this latter effect that turns out to be dominant. Even though the winning probability is decreasing, the effect on the overall welfare of the group is ambiguous: Highly efficient group members with a low valuation may start to provide effort under higher degrees of complementarity and due to their efficiency raise the overall welfare of the group. The results highlight the importance of accounting for within-group heterogeneity and complementarity for a proper analysis of the provision of group-specific public goods in a contest environment.

While these results are derived for the public good "winning probability in a contest", it may be interesting to see whether they hold in general for the private

[3]E.g. Hirshleifer (1983) argues for the special case of perfect complements ("weakest-link" technology) that the complementarity between group members' efforts helps solving the free-rider problem.

[4]Cornes and Hartley (2007) have analyzed a voluntary-contributions to a public-good game with CES production (social-composition) functions where a single group jointly produces a public good. The additional dimension of generality from the contest structure comes at the cost of a more restrictive class of utility functions. Whereas Cornes and Hartley (2007) need binormal utility functions, we assume that utility functions are additively separable between the group-specific public good and some numéraire good that finances individual contributions.

provision of public goods. So far it has not been possible to analyze this point in public goods models such as Cornes and Hartley (2007), since there exist no analytical solutions for the equilibrium contributions aside from some special cases. The fact that in our model we can explicitly solve for equilibrium strategies enables us to perform this comparative-static analysis with respect to the degree of complementarity in efforts.

The paper is organized as follows. We introduce the model in Section 2 and start with introductory examples in Section 3. We characterize the simultaneous Nash equilibria of the general model in Section 4. In section 5 the comparative-static results are summarized. Section 6 concludes. Large proofs are given in the appendix, and in a special Appendix C we will state convergence results for $\vec{\gamma}$ approaching 1, 0, and $-\infty$.

2 The model

Assume that $n \geq 2$ groups compete for a given rent. The set of groups is given by $N = \{1, 2, \ldots, n\}$ while m_i is the number of individuals in group i and k is the index of a generic member of this group. The rent is a group-specific public good that has a value $v_i^k > 0$ to individual k of group i. p_i represents the probability of group $i = 1, ..., n$ to win the contest. It is a function of some vector of aggregate group output $q_1, ..., q_n$. We focus on Tullock-form contest-success functions where the winning probability of a group i is defined as:

Assumption 1: $p_i(Q_1, ..., Q_n) = \begin{cases} \frac{Q_i}{\sum_{j=1}^{n} Q_j}, & \exists Q_j > 0 \\ \frac{1}{n}, & Q_j = 0 \ \forall \ j \end{cases}, i = 1, ..., n$

The aggregate group output Q_i depends on individual effort x_i^k, $Q_i = q_i(x_i^1, ..., x_i^{m_i})$, $i = 1, ..., n$. Following the literature we will call $q_i(.)$ *impact functions* in the following and make the assumption that they are of the constant elasticity of substitution (CES) type.

Assumption 2: $q_i(x_i^1, ..., x_i^{m_i}) = g_i \cdot \left(\sum_{l=1}^{m_i} a_i^l \cdot (x_i^l)^{\gamma_i}\right)^{1/\gamma_i}$, $\gamma_i \in \{(-\infty, 0), (0, 1)\}$, $i = 1, ...n$.

The function has the usual parameters a_i^k for the efficiency of an individual's effort and g_i for the relative strength of the group. Note that we obtain a closed-form solution only if for all i it holds that $\gamma_i \neq 0$. The Cobb-Douglas case $\gamma_i \to 0$ will be covered by a limit result in Appendix C. Also, if $\gamma_i < 0$ and $\exists k : x_i^k = 0$, the function is not well defined. We will therefore take the limit of $q_i(\dots)$ as $x_i^k \to 0$, which means $q_i(\dots) = 0$ in that case. Note that for $\gamma_i > 0$ this is not the case.

> **Assumption 3:** Individuals are risk neutral, face linear costs, and maximize their net rent.

It follows from Assumptions 1, 2, and 3 that the individual expected utility functions are as follows:

$$\pi_i^k(x_1^1, ..., x_n^{m_n}) := \pi_i^k(x_i^k, \vec{x}_{/x_i^k}) = v_i^k \cdot \frac{g_i \cdot \left(\sum_l a_i^l \cdot \left(x_i^l\right)^{\gamma_i}\right)^{1/\gamma_i}}{\sum_j g_j \cdot \left(\sum_l a_j^l \cdot \left(x_j^l\right)^{\gamma_j}\right)^{1/\gamma_j}} - x_i^k, \tag{1}$$

where $\vec{x}_{/x_i^k}$ refers to the vector $x_1^1, ..., x_n^{m_n}$ without x_i^k. We are looking for a Nash equilibrium of this game where individuals choose their effort x_i^k simultaneously to maximize their expected utility,

$$x_i^{k*} \in \arg\max_{x_i^k} \pi_i^k \left(x_i^k, \vec{x}^*_{/x_i^k}\right) \quad \forall i, k, \tag{2}$$

where "$*$" refers to equilibrium values.

3 Introductory examples

In this section we analyze two simple special cases that provide intuition for the relevance of the degree of complementarity in contests. As we will see, the degree of complementarity is only relevant if the valuations between members of the same group differ. The examples restrict attention to a contest between two groups, 1 and 2, with m_1 and m_2 members whose valuations can take two values. The valuation of the group members are either high $\overline{v_i}$ or low $\underline{v_i}$, thus $\overline{v_i} \geq \underline{v_i}$, $i = 1, 2$. The examples are chosen to highlight the central mechanisms of this model, we therefore relegate all technical details about the existence of interior solutions, active and inactive groups and group members, etc. to the next section.

Example 1: Let us restrict attention to groups of equal size $m_1 = m_2 = m$ with only a single valuation of the members of a given group, $\underline{v_i} = \overline{v_i} = v_i, i = 1, 2$ and identical technologies with $a_i^k = 1$, $g_i = 1$ and $\gamma_i = \gamma$. In this case

$$x_1^*(v_1, v_2, m) = \frac{(v_1)^2 \cdot v_2}{m \cdot (v_1 + v_2)^2}, \quad x_2^*(v_1, v_2, m) = \frac{v_1 \cdot (v_2)^2}{m \cdot (v_1 + v_2)^2}$$

constitutes an interior equilibrium. Investments in the contest are independent of γ_i. This example shows that the elasticity of substitution does not play a role if there is no within-group heterogeneity and groups are of equal size and have the same impact function. The reason for this result is the combination of a constant-return to scale impact function with a contest success function that is homogeneous of degree zero. Conversely, it must be either within-group heterogeneity and/or differences in group size and technology that may cause behavioral changes due to changes in γ_i. The next example shows that this may in fact be the case.

Example 2: Let us assume again $m_1 = m_2 = m$ and for all i and k that $a_i^k = 1$ and $\gamma_i = \gamma$. However, we allow for heterogeneous valuations within groups: $\underline{v_1} = \underline{v_2} = \underline{v} \leq \overline{v} = \overline{v_1} = \overline{v_2}$. The population of each group is divided into $\overline{m} = \underline{m} = m/2$ of individuals with the high and the low valuation, respectively. One gets the following symmetric equilibrium:

$$\overline{x}^*(\underline{v}, \overline{v}, m, \gamma) = \frac{\overline{v}}{2 \cdot m \cdot \left((\underline{v}/\overline{v})^{\frac{\gamma}{1-\gamma}} + 1 \right)},$$
$$\underline{x}^*(\underline{v}, \overline{v}, m, \gamma) = \frac{\underline{v}}{2 \cdot m \cdot \left((\overline{v}/\underline{v})^{\frac{\gamma}{1-\gamma}} + 1 \right)}. \quad (3)$$

where $\overline{x}^*$ and $\underline{x}^*$ are the respective equilibrium efforts of the individuals with the high and low valuation. As expected, γ may influence the outcome of the game if differences among the valuations of the rent among the group members exist.

4 The general case

We now turn to the analysis of the general case. In order to have a lean notation, let $X_i = \sum_k x_i^k$, $y_i^k = (x_i^k)^{\gamma_i}$, and $Y_i = (\sum_l a_i^l \cdot y_i^l)$. Further, $Q = \sum_j Q_j = \sum_j g_j \cdot Y_j^{\frac{1}{\gamma_j}} = g_i \cdot Y_i^{\frac{1}{\gamma_i}} + \sum_{j \neq i} g_j Y_j^{\frac{1}{\gamma_j}} = Q_i + Q_{/i}$ in the following. Also, let $\vec{\gamma}$ denote the vector of all γ_i. While deriving the equilibrium strategies, we will omit the parameters of these functions for better readability (e.g y_i^k instead of $y_i^k(\gamma_i, x_i^k)$).

Hillman and Riley (1987) and Stein (2002) have shown that individuals may prefer to stay inactive in a single player contest. Baik (2008) has shown for $\gamma_i = 1$ that only group members with maximum valuation participate in a contest. Hence, it is possible that some individuals and/or groups will stay inactive in our setup. We therefore start with an analysis of active individuals and groups.

Definition 1: An individual k of group i is said to participate if $x_i^k > 0$. A group i is said to participate if there exists some k such that $x_i^k > 0$. A group is said to fully participate if $\forall k : x_i^k > 0$.

Lemma 1: a) *In a Nash equilibrium of a contest fulfilling Assumptions 1, 2, and 3 if a group participates, it fully participates.*
b) If $\gamma_i < 0$, $m_i > 1$, and one group member of group i does not participate, it is always a best response for all group members to not participate.

The proof of this as well as the next Lemma can be found in the appendix. Lemma 1 a) implies that in order to determine whether an individual participates, it is sufficient to determine whether its group participates. Lemma 1 b) shows that irrespective of the behavior of the other groups, it may occur that a group does not participate. The reason is that for $\gamma_i < 0$, a positive contribution from each member is indispensable: As soon as some group member k chooses $x_i^k = 0$, we have $q_i(\dots) = 0$. This of course gives rise to multiple equilibria as a group i may either coordinate on *not participating* or *fully participating* if $\gamma_i < 0$. In the following, we will therefore use the notation that there are $\underline{n} \leq n$ groups with $\gamma_i < 0$ and $m_i > 1$ and we will denote their set as $\underline{N}$, which may be empty. For each of the equilibria determined we must specify a subset of $\underline{N}$ of groups that coordinate on not participating, which will be denoted $\underline{N_0}$.

Let $V_i(\gamma_i) \equiv g_i \cdot \left(\sum_l a_i^l \cdot (a_i^l \cdot v_i^l)^{\frac{\gamma_i}{1-\gamma_i}} \right)^{\frac{1-\gamma_i}{\gamma_i}}$. Without loss of generality, suppose that all groups are ordered with descending V_i such that $V_i(\gamma_i) \geq V_{i+1}(\gamma_{i+1})$. $Q_i^*(\vec{\gamma}, \underline{N_0})$ and $Q^*(\vec{\gamma}, \underline{N_0})$ shall denote Q_i and Q in an equilibrium where $\underline{N_0}$ do not participate. We use $|\dots|$ to denote the cardinality of a set of groups. The following Lemma determines the groups that participate in the equilibrium in which $\underline{N_0}$ do not participate.

Lemma 2: *a) The best response conditions of the members of a group $i \in N/\underline{N_0}$ can be fulfilled, if and only if the following group best response function is fulfilled:*

$$\hat{Q}_i(\vec{\gamma}, Q_{/i}) = \max\left(0, \sqrt{Q_{/i} \cdot V_i(\gamma_i)} - Q_{/i}\right). \tag{4}$$

where $Q_{/i} > 0$ must hold.
b) Groups $N^*(\vec{\gamma}, \underline{N_0}) = \{i \in N/\underline{N_0} : i \leq n^*(\vec{\gamma}, \underline{N_0})\}$ participate, where $n^*(\vec{\gamma}, \underline{N_0}) \equiv$ $\arg\max_{i \in N/\underline{N_0}} i$ *such that $V_i(\gamma_i) > Q^*(\vec{\gamma}, \underline{N_0})$.*
c) Holding $\underline{N_0}$ fixed, if the resulting Nash equilibrium is unique, $Q_i^*(\vec{\gamma}, \underline{N_0})$ and $Q^*(\vec{\gamma}, \underline{N_0})$ are continuous functions in $\vec{\gamma}$.

Lemma 2 a) gives the necessary and sufficient condition for existence of best response strategies of those groups that do not belong to $\underline{N_0}$, for which it is automatically fulfilled. If (4) is not fulfilled, there will be at least one individual who does not play best responses if the group reaches impact $Q_i \neq \hat{Q}_i(\dots)$.

Lemma 2 b) characterizes the participating groups given that the groups in $\underline{N_0}$ do not participate. There are therefore two reasons why a group might not participate: Either because the average valuation of the group members are too low or because it belongs to $\underline{N_0}$. However, once $\underline{N_0}$ is fixed, one can uniquely identify the remaining groups which do not participate.

Lemma 2 c) is useful for the comparative-static analysis if one focuses on a specific equilibrium with given $\underline{N_0}$. Given that the number and identity of active groups then still depends on $\vec{\gamma}$, it is *a priori* not clear that aggregate effort and indirect utilities are continuous in $\vec{\gamma}$. The Lemma reveals that continuity is in fact guaranteed if the identity of groups in $\underline{N_0}$ remains the same. The economic intuition is as follows: Excluding groups among $\underline{N_0}$, assume that $\hat{\gamma}_j$ is a point where a formerly active group becomes inactive or a formerly inactive group becomes active. The aggregate group effort of the active group is continuously reduced to zero as γ_j approaches $\hat{\gamma}_j$, and the formerly inactive group continuously increases its effort from 0 as γ_j increases from $\hat{\gamma}_j$. Hence, there is a "smooth" fade out or fade in of groups at those points.

The following proposition characterizes the set of Nash equilibria of the game. For readability, the strategies x_i^k are defined as functions of $Q^*(\vec{\gamma})$ and $V_i(\gamma_i)$.

Proposition 1. *The set of Nash equilibria of the game characterized by Assumptions 1,2, and 3 is given as follows. For each set of groups in* $\underline{N_0}$ *such that* $|N/\underline{N_0}| \geq 2$ *there exists a Nash equilibrium given by strategies* $x_i^{k*}(\vec{\gamma}, \underline{N_0})$ *that fulfill*

$$x_i^{k*}(\vec{\gamma}, \underline{N_0}) = \begin{cases} Q^*(\vec{\gamma}, \underline{N_0}) \cdot \left(1 - \frac{Q^*(\vec{\gamma}, \underline{N_0})}{V_i(\gamma_i)}\right) \cdot (g_i)^{\frac{\gamma_i}{1-\gamma_i}} \cdot \frac{(a_i^k \cdot v_i^k)^{\frac{1}{1-\gamma_i}}}{V_i(\gamma_i)^{\frac{1}{1-\gamma_i}}}, & i \in N^*(\vec{\gamma}, \underline{N_0}) \\ 0, & else \end{cases}, \tag{5}$$

where $Q^*(\vec{\gamma}, \underline{N_0}) = \frac{|N^*(\vec{\gamma}, \underline{N_0})|-1}{\sum_{i \in N^*(\vec{\gamma}, \underline{N_0})} V_i(\gamma_i)^{-1}}$ *and* $N^*(\vec{\gamma}, \underline{N_0})$ *is defined in Lemma 2 a).*

Proof. Suppose $\underline{N_0}$ do not participate. From Lemma 1 b) we then know that the members of these groups play best responses. Lemma 2 b) determines the participating groups. To obtain $Q^*(\vec{\gamma}, \underline{N_0})$ we sum (4) over all $i \in N^*(\vec{\gamma}, \underline{N_0})$:

$$Q^*(\vec{\gamma}, \underline{N_0}) = \frac{|N^*(\vec{\gamma}, \underline{N_0})| - 1}{\sum_{i \in N^*(\vec{\gamma}, \underline{N_0})} V_i(\gamma_i)^{-1}}. \tag{6}$$

With an explicit solution for $Q^*(\vec{\gamma}, \underline{N_0})$, we can now determine individual expenditures $x_i^{k^*}(\vec{\gamma}, \underline{N_0})$ by solving equation (4) using (6). The participation condition of a group is given by Lemma 2, while Lemma 1 a) ensures that there does not exist an incentive for any group member to deviate to $x_i^k = 0$ in the participating groups. It was further shown that the first-order conditions return local maxima. Since the system of equations given by the first-order conditions of the participating groups has a unique solution this is indeed the unique Nash equilibrium given $\underline{N_0}$. Notice if $N/\underline{N_0}$ has cardinality 1, we have $Q_{/i} = 0$ in (4) and thus best responses are no longer well defined for the participating group. There may be therefore some $\underline{N_0}$ for which no equilibrium exists. However, there is always at least one Nash equilibrium for $\underline{N_0} = \emptyset$. □

Several things are noteworthy: Given the set $\underline{N_0}$, the equilibrium is unique if it exists. Therefore, the maximum number of equilibria is the number of possible combinations of $\underline{N_0}$ such that in total either no groups or at least two groups participate. However, it is possible that some of these equilibria are identical since removing group i from $\underline{N_0}$ does not necessarily mean that it enters N^*.

Further, it may be a Nash equilibrium that no group participates if for all i we have $\gamma_i < 0$ and $m_i > 1$. It also may occur that for some $\underline{N_0}$ no Nash equilibrium exists, since for (5) to be well defined it is required that at least two groups participate. An $\underline{N_0}$ that leaves only one potentially participating group will therefore not yield a Nash equilibrium.

A focal special case has no intra-group heterogeneity $v_i^k = v_i \forall k \forall i$ and $a_i^k = a_i \forall k \forall i$. The following corollary of Proposition 1 can then be established.

Corollary 1: *Let $\underline{N_0} = \emptyset$. Suppose for all groups i and all individuals k, it holds that $a_i^k = a_i$ and $v_i^k = v_i$ and further for all other groups j it holds that $a_i \cdot m_i = a_j \cdot m_j$. Then the equilibrium efforts are independent of $\vec{\gamma}$.*

Proof. Inserting the above values for every individual l $a_i^l = a_i$ and $v_i^l = v_i$ and setting for all other groups j $a_j \cdot m_j = a_i \cdot m_i$ into (5) directly yields the result. □

The corollary shows that $\vec{\gamma}$ is only relevant if there is either heterogeneity with respect to valuations within groups and/or heterogeneity with respect to group size. In all other cases equilibrium behavior does not depend on $\vec{\gamma}$ with the exception that groups may fall into the set $\underline{N_0}$ if their γ_i drops below 0. The corollary shows that corresponding results from public-goods games with complementarities in efforts (Cornes (1993); Cornes and Hartley (2007)) continue to hold in a contest environment. This finding implies that an increase in complementarity between group members' effort *per se* has no effect on the within-group free-rider problem, as could have been conjectured from Hirshleifer (1983). A further implication of the corollary is that the results on group contests that have been derived in the literature for the case of perfect substitutes or perfect complements carry over to arbitrary elasticities of substitution if homogeneous groups differ only in their valuations of the rent and their group efficiency parameter g_i.

5 Comparative statics

Before we move on to the comparative-static analysis, let us first note that the winning probability of group i takes the form:

$$p_i(Q_1^*(\vec{\gamma}), ..., Q_n^*(\vec{\gamma})) = \frac{{Q_i}^*(\vec{\gamma})}{Q^*(\vec{\gamma})} = \left(1 - \frac{Q^*(\vec{\gamma})}{V_i(\gamma_i)}\right), \tag{7}$$

which can be derived from (4). An analysis of convergence results for γ_i which can be found in Appendix C suggests that it makes sense to generally impose $\sum_k a_i^k = 1$ to model relative differences in efficiency between group members and use the parameter g_i for the resulting absolute differences in efficiency between groups. Only then the comparative statics with respect to $\vec{\gamma}$ will capture solely the effect of different degrees of substitution and no productivity effects.

We now turn to the comparative-static analysis of the influence of the elasticity of substitution on the behavior in the contest using the approach developed by Cornes and Hartley (2005). Since we have multiple equilibria for $\gamma_i < 0$, it is necessary to exclude jumps from one equilibrium to another. We will therefore focus in the following on the equilibrium given by some $\underline{N_0}$ in which at least two groups participate.

Most interestingly, individual valuations in relation to the valuations of the other group members define the individuals' share of the amount of effort spent by the group, x_i^{k*}/X_i^*. The valuation of other groups have no effect on these shares. As was to be expected, a larger elasticity of substitution γ_i increases *ceteris paribus* the dispersion of these shares, since in equilibrium the exponent discriminates more strongly between differences in (efficiency-weighted) valuations. The next proposition states the effect of γ_i on the individual shares.

Proposition 2. *Suppose group i participates. The share of an individual of its group's effort,* $\frac{x_i^k}{X_i}$*, increases (decreases) strictly in the elasticity of substitution among efforts if the valuation times the efficiency* $a_i^k \cdot v_i^k$ *of the individual is strictly larger (smaller) than the share-weighted geometric mean of the group members' valuation times efficiency,* $\prod_l (a_i^l \cdot v_i^l)^{\left(\frac{x_i^l}{X_i}\right)}$.

Proof. It is straightforward to derive the following equation from (5):

$$\frac{x_i^{k*}(\vec{\gamma})}{X_i^*(\vec{\gamma})} = \frac{(a_i^k \cdot v_i^k)^{\frac{1}{1-\gamma_i}}}{\sum_l (a_i^l \cdot v_i^l)^{\frac{1}{1-\gamma_i}}} \tag{8}$$

Taking the derivative of (8) with respect to γ_i yields

$$\frac{\partial \frac{x_i^k}{X_i}}{\partial \gamma_i} = \frac{(a_i^k \cdot v_i^k)^{\frac{1}{1-\gamma_i}}}{\sum_l (a_i^l \cdot v_i^l)^{\frac{1}{1-\gamma_i}}} \frac{1}{(1-\gamma_i)^2} \left(\ln(a_i^k \cdot v_i^k) - \frac{\sum_l (a_i^l \cdot v_i^l)^{\frac{1}{1-\gamma_i}} \ln(a_i^l \cdot v_i^l)}{\sum_l (a_i^l \cdot v_i^l)^{\frac{1}{1-\gamma_i}}} \right). \tag{9}$$

The RHS of the above equation is positive whenever the term in brackets is positive. Setting $\ln(a_i^k \cdot v_i^k) \geq \sum_l (a_i^l \cdot v_i^l)^{\frac{1}{1-\gamma_i}} \ln(a_i^l \cdot v_i^l) / \sum_l (a_i^l \cdot v_i^l)^{\frac{1}{1-\gamma_i}}$ and rearranging yields the condition:

$$\frac{\partial \frac{x_i^k}{X_i}}{\partial \gamma_i} \gtreqless 0 \quad \Leftrightarrow \quad a_i^k \cdot v_i^k \gtreqless \prod_l (a_i^l \cdot v_i^l)^{\left(\frac{(a_i^l \cdot v_i^l)^{\frac{1}{1-\gamma_i}}}{\sum_s (a_i^s \cdot v_i^s)^{\frac{1}{1-\gamma_i}}} \right)}. \tag{10}$$

□

The proposition implies that for all group members with a valuation above the weighted geometric mean, the share of total group effort increases with γ_i. The result shows that the dispersion of valuations plays a crucial role for the comparative-static effects of γ_i. In the easiest case of a two-member group i with individuals j and k, the proposition boils down conveniently: Individual j's share increases in γ_i if and only if $a_j v_j > a_k v_k$: The individual with the higher efficiency-weighted valuation increases its relative contributions if γ_i goes up. In the context of the partnership example from the introduction, the finding implies that the relative burden for group success is increasingly carried by the individuals with either the highest stakes and/or the highest productivity if it becomes easier to substitute between the partners' efforts. The reason is as follows. A higher elasticity of substitution has two effects. From the point of view of the high-stake / high productivity player, effort becomes less dependent on the other players' efforts, which *ceteris paribus* gives an additional stimulus to invest relatively more. And from the point of view of his fellow team mates, the negative effects of slacking off become less detrimental, which *ceteris paribus* implies that it pays to invest relatively less.

A second interesting question may be whether the winning probability of groups can be increased by a higher degree of complementarity of efforts. The intuition behind this may be twofold: First, with higher complementarity, the free-rider problem is solved better, such that also individuals with low valuations participate. Second, there often exist gains from specialization. While the latter intuition is induced by the technology itself, which is exogenous in our model, the first intuition can be examined through comparative statics of the model.

Proposition 3. *Suppose $\sum_k a_i^k = 1$. Then the winning probability of a participating group i is weakly increasing in γ_i and strictly increasing whenever there exist two group members k and l such that $a_i^k \cdot v_i^k \neq a_i^l \cdot v_i^l$ and the change in γ_i does not turn any group from a participative into a non-participative status.*

Proof. Using (6), the winning probability of group i, (7), can be written as

$$\frac{{Q_i}^*(\vec{\gamma}, \underline{N_0})}{Q^*(\vec{\gamma}, \underline{N_0})} = \left(1 - \frac{n^*(\vec{\gamma}) - 1}{1 + \sum_{j \neq i} \frac{V_i(\gamma_i)}{V_j(\gamma_j)}}\right), \tag{11}$$

where the sum refers to all active groups $1, ..., n^*(\vec{\gamma})$ except i. Two cases have to be distinguished: (a) A change in γ_i turns group i from a participative to a non-participative status or leaves its non-participative status intact. In this case, the change in γ_i has no influence on group i's winning probability because of the smooth fade out of the group's investments. (b) A change in γ_i has no influence on the participative status of i. In this case, note that (11) is strictly increasing in $V_i(\gamma_i)$. $V_i(\gamma_i)$ has under the assumption of $\sum_k a_i^k = 1$ the structure of an a_i^k weighted power mean of the $a_i^k \cdot v_i^k$ values of the group members. By the weighted power mean inequality (Bullen (2003)) we know that $V_i(\gamma_i)$ is strictly increasing in $\vec{\gamma}$ whenever there exist two individuals with $a_i^k \cdot v_i^k \neq a_i^l \cdot v_i^l$. Whenever all individuals have the same $a_i^k \cdot v_i^k$, $V_i(\gamma_i) = g_i \cdot a_i^k \cdot v_i^k$ and is therefore independent of γ_i. □

This result contradicts the common intuition that higher complementarity leads to a better solution of the free-rider problem and thus a better performance of the group. The result shows exactly the opposite: All things equal, heterogeneous groups with higher complementarity perform worse than similar groups with low complementarity. The intuition behind this is that a lower γ_i puts more emphasis on the lower values of x_i^k, so the lower γ_i, the more the equilibrium will reflect the optimal q_i of lower valuation group members. This has an important implication for the provision of public goods by groups in general: Highly complementary technologies will only be used if there are sufficient gains of specialization coming with them. While higher complementarity solves the free-rider problem, it solves it in the worst possible way: By reducing the incentives of high valuation individuals more than increasing the incentives of low valuation individuals.

For the partnership example from the introduction, Proposition 3 implies that if the production of impact of a partnership becomes more complementary, the equilibrium share or winning probability for this partnership goes down. If as a thought experiment one defines the sum of prizes of the partners in a partnership as total profit, the distribution of these profits depends on the shares of the partners in the firm, and can therefore be considered a design element. If in addition one considers the degree of complementarity also as a design element (because it depends at least to a certain extend on the organizational structure and the business model of the partnership), Proposition 3 reveals a rather odd implication for the share- or winning-probability maximizing design: The partnership would try to minimize complementarities. If it is possible to reach perfect substitutability, it would allocate all the profit shares to the single, most productive and / or highest-stake individual (Olson (1965) and Ray et al. (2007)). This conclusions runs counter to the intuition that complementarity in efforts encourages division of labor. Our finding isolates the *pure* effect of complementarity and shows that this pure effect alone is not only insufficient but counterproductive to explain gains from the division of labor. It is true that the division of labor comes with specialization, which makes individual efforts complementary. But the gains from specialization must result from an increase in group productivity, and this increase must be sufficiently strong to overcompensate the negative effect resulting from an increase in complementarity. If groups cannot use incentive mechanisms to internalize the within-group externalities, a free-rider problem exists for all degrees of complementarity and the effects are the more severe the higher the complementarity.

Our result is also novel in the literature on public-goods games in which no general comparative-static results have been provided for the effect of complementarity in efforts on the provision of public goods for heterogeneous contributors. The fact that we can solve for equilibrium strategies analytically allows us to perform this analysis here. This may also motivate a reexamination of the public-goods games in Cornes (1993) and Cornes and Hartley (2007) outside a contest setting to verify whether this result carries over to other public-goods games. Since it is not generally possible to solve for equilibrium strategies analytically in these models, one can expect this to be a nontrivial task, however.

Given that the winning probability of group i is monotonically increasing in γ_i, we may be interested in whether the same is true for the expected payoff. It turns out that the effect on the expected payoff of the group members is ambiguous both for the aggregate of players as well as the individual players. Inserting (5) into (2) we obtain:

$$\pi_i^k = p_i \left(v_i^k - Q^*(\vec{\gamma}, \underline{N_0})(g_i)^{\frac{\gamma_i}{1-\gamma_i}} \frac{(a_i^k \cdot v_i^k)^{\frac{1}{1-\gamma_i}}}{V_i(\gamma_i)^{\frac{1}{1-\gamma_i}}} \right). \tag{12}$$

As we know from Proposition 3, p_i is increasing in γ_i. However, also $Q^*(\vec{\gamma}, \underline{N_0})$ is increasing in γ_i and for sufficiently high $a_i^k v_i^k$ the term $(g_i)^{\frac{\gamma_i}{1-\gamma_i}} \cdot \frac{(a_i^k \cdot v_i^k)^{\frac{1}{1-\gamma_i}}}{V_i(\gamma_i)^{\frac{1}{1-\gamma_i}}}$ may be increasing as well. Therefore, for the group members with the highest $a_i^k \cdot v_i^k$, expected utility may be decreasing in γ_i. It is also clear that the group members of the lowest $a_i^k \cdot v_i^k$ will always improve their expected payoff by lower complementarity, since they will strictly reduce their effort and the group has a higher winning probability. The optimal γ_i a utilitarian planner who maximizes the sum of the group members' expected payoffs would impose is ambiguous: A lower γ_i may induce individuals with a lower valuation v_i^k but higher efficiency a_i^k to exert higher effort. If the highest type has a high valuation but a low efficiency, this may lead to overall efficiency gains for the group. From a group-production perspective one can understand the underlying mechanism in the following way: By changing the incentives of the group members, different degrees of complementarity also change the shares of effort provided by them. In turn, under heterogeneous technologies, this also changes the shares of total effort used by the different technologies. Different γ_i will therefore not only influence the effort X_i provided by the group, but also the average efficiency of the group in converting this effort into impact. To get a better intuition for this result we turn to an example.

Example 4: Since we are only interested in the effects of higher complementarity for one group, let the aggregate of the valuations of the first group be $V_1(\gamma_1) = 10$. Since this is the only way in which parameters from the first group enter the decision problem of the second, no more information about group one would be necessary. One could for example think of a group of a single individual with $v_1^1 = 10$, $a_1^k = 1$, and $g_1 = 1$. For the second group, assume two individuals with efficiency parameters $a_2^1 = 0.2$ and $a_2^2 = 0.8$. Thus, $\sum_l a_2^l = 1$ and comparative statics over γ_2 contain no effects from changes in productivity. Further, let valuations be heterogeneous such that $v_2^1 = 30$ and $v_2^2 = 5$. Finally, the efficiency parameter of the group is $g_2 = 1$.

From the fact that $v_2^1 \cdot a_2^1 = 6 > 4 = v_2^2 \cdot a_2^2$, we know that for $\gamma_2 = 1$ only the first individual will participate and for $\gamma_2 \to -\infty$, both individuals will participate. Proposition 3 tells us that the winning probability will decrease with lower values of γ_2.

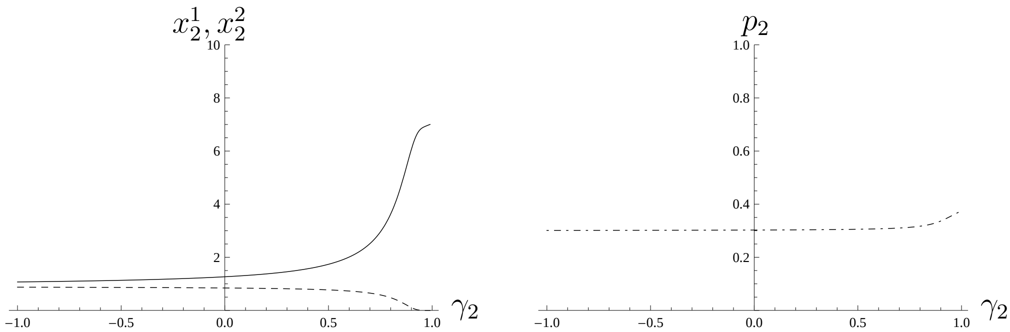

Figure 1: Effort levels and winning probability for different values of γ_2.

From Figure 1 we can see how this translates into our example. The effort level of individual 2 (with high efficiency and low valuation, dashed line) slowly increases as we reduce γ_2, while the effort of individual 1 (solid line) falls. Both converge as $\gamma_2 \to -\infty$. We also see that the winning probability is falling with lower values of γ_2, as expected. The free-rider problem is thus solved with lower γ_2, but in a way such that the overall winning probability of the group is decreased. The more interesting result is, however, how this translates into the expected utility of the individuals.

In Figure 2 we see the expected utility of individuals 1 and 2 (again, represented by solid and dashed lines) and the aggregate expected utility (dotdashed line). The

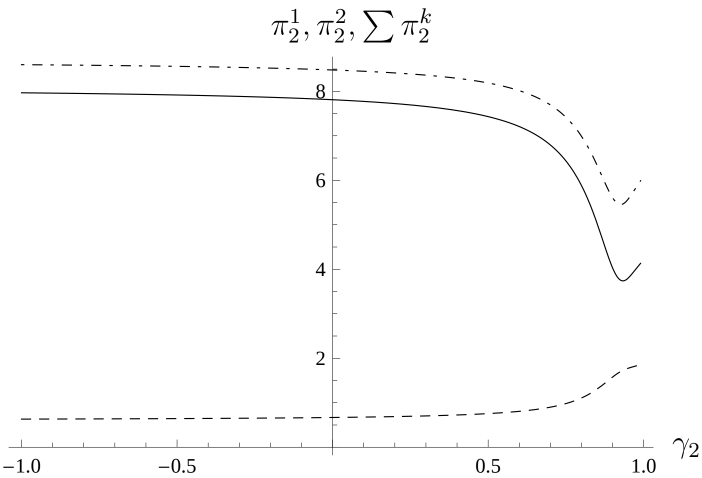

Figure 2: Expected utility for different values of γ_2.

expected utility of individual 2 is of course rising in γ_2 (falling with higher complementarity), since in the case of perfect substitutes, i.e. $\gamma_2 = 1$, individual 2 can fully free ride. The change of expected utility of individual 1 is ambiguous with respect to changes in γ_2. For very high values of γ_2, it is also increasing with γ_2, while for low values it is decreasing in γ_2. Aggregate expected utility is mainly influenced by individual 1 and thus total expected utility of the group members behaves similarly: It is also maximal for very high degrees of complementarity and has a minimum below $\gamma_2 = 1$. The result is driven by the fact that the efficiency of individual 2 is much higher than that of individual 1 and at the same time the valuation of individual 1 is much higher than that of individual 2. In the perfect-substitutes case $\gamma_2 = 1$, only the less efficient individual 1 contributes effort and individual 2 takes a free ride. As we move away from this case, individual 2's incentives to provide effort increase only slowly. Due to the complementarity, individual 1 incurs very high losses in these cases. Reducing γ_2 even further provides much stronger incentives for individual 2. Individual 1 can thus reduce its effort further and in turn gain utility from the higher complementarity.

6 Concluding Remarks

This paper has started from the observation that group effort can in general not be additively decomposed into some sum (of functions) of individual efforts. The use of a CES-impact function has allowed to identify the main channels of influence of the elasticity of substitution on the behavior in and the outcome of contests. If groups have are homogeneous (i.e. all group members have the same valuation and efficiency within the group), the elasticity of substitution does not matter. For heterogeneous groups, the higher the complementarity of efforts of that group, the lower the divergence of efforts among group members and the lower the winning probability of that group. This contradicts the common intuition that groups can improve their performance by solving the free-rider problem via higher degrees of complementarity of efforts. Only if very high valuation members are also very inefficient at effort production the total expected utility may be higher for higher degrees of complementarity: At high levels of complementarity, highly efficient individuals with low valuations may replace some of the effort that is provided by less efficient group members at low levels of complementarity. The beneficial or detrimental role of complementarity for a group is therefore undetermined without further information on the composition of a group.

Appendix A: Proof of Lemma 1

Proof. For the proof of Lemma a) we first check that the interior solution is a local maximum if all group members participate. The first-order condition of the maximization problem (2) can be written as

$$\frac{Q_{/i}}{Q^2} Y_i^{\frac{1}{\gamma_i}-1} = \frac{(y_i^k)^{\frac{1}{\gamma_i}-1}}{v_i^k}. \tag{A.1}$$

The second-order condition is satisfied if

$$\frac{v_i^k \cdot Q_{/i} \cdot Y_i^{\frac{1}{\gamma_i}-2}}{\gamma_i \cdot Q^2} \left(\frac{1 - 2 \cdot \frac{Q_i}{Q}}{\gamma_i} - 1 \right) - \frac{\frac{1}{\gamma_i} - 1}{\gamma_i} \cdot (y_i^k)^{\frac{1}{\gamma_i}-2} < 0. \tag{A.2}$$

Solving the first-order condition for v_i^k and inserting the expression into the second-order condition we obtain, upon rearranging:

$$\frac{1 - \frac{1}{\gamma_i}}{\gamma_i} \left(1 - \frac{y_i^k}{Y_i} \right) - 2 \cdot \frac{1}{\gamma_i^2} \cdot \frac{Q_i \cdot y_i^k}{Q \cdot Y_i} < 0, \tag{A.3}$$

which holds for all $\gamma_i \in \{(-\infty, 0), (0, 1)\}$. Therefore, all solutions of the first-order condition are local maxima taking the other players' strategies as given. The best responses are either given by the solution to the first-order condition, or by a corner solution. From equation (1) it is clear that the only possible corner solutions are non-participation with $x_i^k = 0$. We thus need to verify that whenever the best response of one member of the group is given by the solution to the first-order condition, it is not possible for any member of the group to have the best response $x_i^k = 0$.

First, we will show that whenever there exists a solution of the first-order condition for one individual of a group, it exists for all individuals: From the first-order conditions of two representative group members l, k we obtain the within-group equilibrium condition:

$$\forall l, k: \quad \frac{(y_i^k)^{\frac{1}{\gamma_i}-1}}{v_i^k} = \frac{(y_i^l)^{\frac{1}{\gamma_i}-1}}{v_i^l} \tag{A.4}$$

for all members k, l of group i. Both, the left-hand side (LHS) and right-hand side (RHS) of (A.4) are strictly increasing in y_i^k, y_i^l if $\vec{\gamma} \in (0,1)$. For $\vec{\gamma} \in (-\infty, 0)$ both LHS and RHS of (A.4) are strictly decreasing in y_i^k, y_i^l. Thus, for each y_i^k there exists a y_i^l such that the within-group equilibrium condition holds. Since for all group members the LHS of (A.1) is equal, there exists a positive solution to the first-order condition (FOC) for either all group members or none.

Second, we need to show that $x_i^k = 0$ is not a best response if it is a best response for another individual l in the group to play $x_i^l > 0$. We do so by contradiction: Obviously, for a corner solution with $x_i^k = 0$ and $x_i^l > 0$ the following condition needs to hold:

$$\frac{\partial \pi_i^k}{\partial x_i^k} = \frac{Q_{/i}}{Q^2} \cdot Y_i^{\frac{1}{\gamma_i}-1} \cdot (x_i^k)^{\gamma_i - 1} \cdot v_i^k - 1 \Big|_{x_i^k = 0, x_i^l > 0} \leq 0. \tag{A.5}$$

From the fact that there is an individual l in the group, which participates with strictly positive effort, we know that

$$\frac{\partial \pi_i^l}{\partial x_i^l} = \frac{Q_{/i}}{Q^2} \cdot Y_i^{\frac{1}{\gamma_i}-1} \cdot (x_i^l)^{\gamma_i - 1} \cdot v_i^l - 1 \Big|_{x_i^k = 0, x_i^l > 0} = 0. \tag{A.6}$$

Inserting (A.6) into (A.5) yields:

$$\frac{(x_i^l)^{1-\gamma_i}}{v_i^l} - \frac{(x_i^k)^{1-\gamma_i}}{v_i^k} \Big|_{x_i^k = 0, x_i^l > 0} \leq 0 \tag{A.7}$$

from which we obtain by inserting $x_i^k = 0$:

$$(x_i^l)^{1-\gamma_i} \Big|_{x_i^l > 0} \leq 0 \tag{A.8}$$

which is a contradiction for all $\gamma_i < 1$. Thus there does not exist an equilibrium in which for one player in the group a corner solution at zero effort investments is obtained while for another an interior solution holds.

Part b) can be shown as follows: Suppose $x_i^k = 0$ for some k, $m_i \geq 2$ and $\gamma_i < 0$. Then $q_i(x_i^1, \ldots, x_i^{m_i}) = 0$. The expected payoff $\pi_i^l(x_i^1, \ldots, x_n^{m_i})$ of any other group member is then strictly decreasing in its own effort x_i^l independent of $Q_{/i}$. Therefore, $x_i^k = 0$, $x_i^l = 0$ are mutually best responses for all group members, independent of the behavior of other groups reflected in $Q_{/i}$.

□

Appendix B: Proof of Lemma 2

Proof. Suppose $i \notin \underline{N_0}$. If there exists a solution to the FOC, it is characterized by the following equation, obtained by solving (A.4) for y_i^l and summing over all l,

$$Y_i = y_i^k \cdot \sum_l (\frac{v_i^l}{v_i^k})^{\frac{\gamma_i}{1-\gamma_i}}. \tag{B.1}$$

We can now solve equation (A.1) for Y_i explicitly:

$$Y_i = \left(\sqrt{Q_{/i} \cdot V_i(\gamma_i)} - Q_{/i}\right)^{\gamma_i}. \tag{B.2}$$

Thus, the condition for a strictly interior solution is $(\sum_l v_i^{l\,\frac{\gamma_i}{1-\gamma_i}})^{\frac{1-\gamma_i}{\gamma_i}} > Q_{/i}$. Note that this condition is the same for all members of a group. In all other cases, we get $y_i^k = 0$ for $\gamma_i \in (0, 1)$ and $y_i^k = \infty$ for $\gamma_i \in (-\infty, 0)$ as was to be expected and which corresponds to $x_i^k = 0$. In these cases we have $\forall l : y_i^k = y_i^l$ by equation (A.4) and by the definition of Q_i, we have: $Q_i = Y_i^{\frac{1}{\gamma_i}} = 0$. We can write a group best-response function as

$$\hat{Q}_i(\gamma_i, Q_{/i}) = \max\left(0, \sqrt{Q_{/i} \cdot V_i(\gamma_i)} - Q_{/i}\right). \tag{B.3}$$

establishing part a), since by Lemma 1 either for all group members we obtain an interior solution or for none. Since the best-response function is continuous in $\gamma_i \neq 0$ and in the strategies of the other groups $Q_{/i}$, if a unique Nash equilibrium exists, the equilibrium strategies must also be continuous in all γ_i. This establishes part c) of Lemma 2. What remains to be shown is which groups participate in equilibrium given that $\underline{N_0}$ do not participate. Suppose a group ζ participates in equilibrium with strictly positive effort, while a group $\zeta + 1$ does not participate. Let $Q_i^*(\vec{\gamma}, \underline{N_0})$ be Q_i in equilibrium (the notation ignores here that these are best responses and should thus be functions of $Q_{/i}^*$) and let the other variables introduced above be defined correspondingly in equilibrium. Then by the above condition in equilibrium we have for any given $\vec{\gamma}$:

$$V_\zeta(\gamma_i) > Q^*_{/\zeta}(\vec{\gamma}, \underline{N_0})$$
$$V_{\zeta+1}(\gamma_i) \leq Q^*_{/\zeta+1}(\vec{\gamma}, \underline{N_0}) \tag{B.4}$$

Since by assumption $Q^*_{\zeta+1}(\vec{\gamma}, \underline{N_0}) = 0$, we have $Q^*_{/\zeta+1}(\vec{\gamma}, \underline{N_0}) = Q^*(\vec{\gamma}, \underline{N_0})$. Solving (4) for $Q_{/i}$ tells us that in an equilibrium where group ζ participates, the following needs to be true:

$$Q^*_{/\zeta}(\vec{\gamma}, \underline{N_0}) = \frac{Q^*(\vec{\gamma}, \underline{N_0})^2}{V_\zeta(\gamma_i)}. \tag{B.5}$$

We now insert (B.5) into the first equation of (B.4) and the condition $\hat{Q}_{/\zeta+1} = \hat{Q}$ into the second equation. Thus, the condition (B.4) becomes

$$V_\zeta(\gamma_\zeta) > Q^*(\vec{\gamma}, \underline{N_0})$$
$$V_{\zeta+1}(\gamma_{\zeta+1}) \leq Q^*(\vec{\gamma}, \underline{N_0}) \tag{B.6}$$

in equilibrium. It follows that $V_\zeta(\gamma_\zeta) > V_{\zeta+1}(\gamma_{\zeta+1})$. We can thus order the groups such that $V_i(\gamma_i) \geq V_{i+1}(\gamma_{i+1})$ and define $n^*(\vec{\gamma}, \underline{N_0})$ as the group with the highest index number in $N/\underline{N_0}$ that still participates with strictly positive effort. By (B.6), all groups with $i \in N/\underline{N_0}$ and $i \leq n^*(\vec{\gamma})$ participate. This establishes part b) of Lemma 2. □

Appendix C: Convergence Results

We will now state convergence results where for all groups j, γ_j approaches 1, 0, and $-\infty$. X_i^* denotes X_i in equilibrium. Throughout we will assume $\underline{N_0} = \emptyset$.

Proposition 4. *For $\gamma_i \to 1^-$, we get $\frac{x_i^{k*}}{X_i^*} = 0$ if $\exists a_i^l v_i^l > a_i^k v_i^k$ and $\frac{1}{\sharp\{l : a_i^l \cdot v_i^l = a_i^k \cdot v_i^k\}}$ otherwise.*

Proof. It is straightforward to derive the following equation from (5):

$$\frac{x_i^{k*}(\vec{\gamma})}{X_i^*(\vec{\gamma})} = \frac{(a_i^k \cdot v_i^k)^{\frac{1}{1-\gamma_i}}}{\sum_l (a_i^l \cdot v_i^l)^{\frac{1}{1-\gamma_i}}} \tag{C.1}$$

For the limit it then holds:

$$\lim_{\gamma_i \to 1} \frac{(a_i^k \cdot v_i^k)^{\frac{1}{1-\gamma_i}}}{\sum_l (a_i^l \cdot v_i^l)^{\frac{1}{1-\gamma_i}}} = \lim_{\gamma_i \to 1} \left(\sum_l \left(\frac{a_i^l \cdot v_i^l}{a_i^k \cdot v_i^k} \right)^{\frac{1}{1-\gamma_i}} \right)^{-1}$$
$$= \begin{cases} 0, & \exists a_i^l \cdot v_i^l > a_i^k \cdot v_i^k \\ \frac{1}{\sharp\{a_i^l \cdot v_i^l : a_i^l \cdot v_i^l = a_i^k \cdot v_i^k\}}, & else \end{cases} . \qquad \text{(C.2)}$$

□

Proposition 4 shows that for γ_i increasing towards one, the group members with lower valuations will decrease their efforts towards zero, and only the group members with the highest valuations contribute. If there is more than one individual with the highest valuation, we converge to an equilibrium where those individuals contribute equally. In this case we get multiple equilibria if $\gamma_i = 1$ with the property that the sum of contributions is always identical (Baik (2008)). In this sense, our convergence result can be interpreted as an equilibrium-selection mechanism which selects the equal-contributions equilibrium from the multiple equilibria in Baik (2008).

Next we will analyze the other boundary case when all γ_j approach $-\infty$. In order to have a lean notation we denote $\gamma_j = \gamma$ and $\lim_{\gamma \to -\infty} f(\gamma)$ by $f(-\infty)$ for all functions $f(.)$:

Proposition 5. *For $\gamma \to -\infty$, we obtain:*
a) $\lim_{\gamma \to -\infty} V_i(\gamma) = \frac{g_i}{m_i} HM(v_i^1, \ldots, v_i^{m_i})$
b) $\lim_{\gamma \to -\infty} \frac{x_i^{k*}(\gamma)}{X_i^*(\gamma)} = \frac{1}{m_i}$
c) $\lim_{\gamma \to -\infty} Q^*(\gamma) = \frac{n^*(-\infty)}{\sum_j \sum_l 1/(v_j^l \cdot g_i)}$
d) x_i^{k*} *is independent of* a_j^l $\forall j, l$
where $HM(v_i^1, \ldots, v_i^{m_i}) = \frac{m_i}{\sum_l \frac{1}{v_i^l}}$ *is the harmonic mean of the valuations within the group.*

The results follow directly from the determination of the limit of (5).

Since relative strength of groups is determined by V_i, the limit behavior of V_i is of course of great interest. From Proposition 5 b) we see that the distribution and level of relative strengths a_i^k of each group member have no effect on V_i. The irrelevance of a_i^k is further shown by part d) of the proposition, where we see that even equilibrium efforts x_i^{k*} are unaffected by a_i^k. This was to be expected, since under perfect complements in fact all inputs are crucial for the level of q_i. Proposition

5 b) shows that (as expected given the results by Lee (2012)) all group members participate with equal amounts. In this sense, for γ near $-\infty$, we obtain similar results as for a min(...) impact function. However, this function creates multiple equilibria with an associated equilibrium-selection problem. Given the uniqueness of equilibria for all finite $\vec{\gamma}$, our limit result can be interpreted as an equilibrium-selection mechanism where individual contributions depend on the harmonic mean of the valuations.

Next we look at the limit behavior for $\gamma \to 0$. It turns out that we have to consider $\gamma \to 0^+$ and $\gamma \to 0^-$ separately because the problem may not be continuous at this point.

Proposition 6. *At $\gamma_i = 0$, $V_i(\gamma_i)$ is discontinuous if $\sum_l a_i^l \neq 1$.*

$$\lim_{\gamma_i \to 0^+} V_i = \begin{cases} \infty, & \sum a_i^k > 1 \\ \prod (a_i^k \cdot v_i^k)^{a_i^k}, & \sum a_i^k = 1 \\ 0, & \sum a_i^k < 1 \end{cases}, \tag{C.3}$$

$$\lim_{\gamma_i \to 0^-} V_i = \begin{cases} 0, & \sum a_i^k > 1 \\ \prod (a_i^k \cdot v_i^k)^{a_i^k}, & \sum a_i^k = 1 \\ \infty, & \sum a_i^k < 1 \end{cases}. \tag{C.4}$$

Since the winning probability, the equilibrium efforts, and impacts are all functions of all V_i, it follows that these values will in general also be discontinuous in γ_i. In particular, the winning probability and the participation condition of group i are increasing functions of V_i. For $\gamma \to 0^+$ the group with the strictly highest $\sum a_i^k$ will therefore win with probability one while for $\gamma \to 0^-$ the group with the strictly lowest $\sum a_i^k$ will win with probability one. Only if all groups have $\sum a_i^k = 1$, these effects do not occur and we obtain for V_i the a_i^k-weighted geometric mean of $v_i^k \cdot a_i^k$. To obtain a proper intuition for the behavior near $\gamma = 0$, it is helpful to show an example.

Example 3: Assume that $v_1 = v_2$ but allow for differences in group size with $m_i > 1$. Further, we fix $a_i^k = 1$, $g_i = g_j = 1$, and $\gamma_i = \gamma$. Therefore, we are always in a situation with $\sum a_i^k = m_i > 1$. In this case, (5) implies

$$x_1(m_1, m_2, \gamma, v) = \frac{v \cdot m_1^{\frac{1-2\gamma}{\gamma}} \cdot m_2^{\frac{1-\gamma}{\gamma}}}{\left(m_1^{\frac{1-\gamma}{\gamma}} + m_2^{\frac{1-\gamma}{\gamma}}\right)^2},\ x_2(m_1, m_2, \gamma, v) = \frac{v \cdot m_1^{\frac{1-\gamma}{\gamma}} \cdot m_2^{\frac{1-2\gamma}{\gamma}}}{\left(m_1^{\frac{1-\gamma}{\gamma}} + m_2^{\frac{1-\gamma}{\gamma}}\right)^2}, \tag{C.5}$$

in a within-group symmetric equilibrium. In this case, individual efforts depend on the size of the groups. Coming back to Example 2, (C.5) can be used to determine that the values of the impact functions are

$$q_1(m_1, m_2, \gamma, v) = v \cdot \frac{m_1^{\frac{1-\gamma}{\gamma}} \cdot m_2^{\frac{1-\gamma}{\gamma}}}{\left(m_1^{\frac{1-\gamma}{\gamma}} + m_2^{\frac{1-\gamma}{\gamma}}\right)^2}, \quad q_2(m_1, m_2, \gamma, v) = v \cdot \frac{m_1^{\frac{1-\gamma}{\gamma}} \cdot m_2^{\frac{1-\gamma}{\gamma}}}{\left(m_1^{\frac{1-\gamma}{\gamma}} + m_2^{\frac{1-\gamma}{\gamma}}\right)^2},$$

which in turn can be used to determine the equilibrium winning probabilities:

$$p_1(m_1, m_2, \gamma) = \frac{m_2^{\frac{\gamma-1}{\gamma}}}{m_1^{\frac{\gamma-1}{\gamma}} + m_2^{\frac{\gamma-1}{\gamma}}}, \quad p_2(m_1, m_2, \gamma) = \frac{m_1^{\frac{\gamma-1}{\gamma}}}{m_1^{\frac{\gamma-1}{\gamma}} + m_2^{\frac{\gamma-1}{\gamma}}}. \tag{C.6}$$

The limit behavior of these probabilities is

$$\lim_{\gamma \to 0^-} p_1(m_1, m_2, \gamma) = \begin{cases} 1, & m_1 < m_2 \\ 0, & m_1 > m_2 \end{cases},$$

$$\lim_{\gamma \to 0^+} p_1(m_1, m_2, \gamma) = \begin{cases} 0, & m_1 < m_2 \\ 1, & m_1 > m_2 \end{cases},$$

and analogously for $p_2(m_1, m_2, \gamma)$. Figure 3 shows $p_1(m_1, m_2, \gamma)$ (dashed line) and $p_2(m_1, m_2, \gamma)$ (solid line) for the case $m_1 > m_2$. We will focus on $p_1(m_1, m_2, \gamma)$ in the following. The graph starts at 0.5 at $\gamma = 1$. This is the well-known case where group size has no impact on the winning probability (Baik (2008)). $p_1(m_1, m_2, \gamma)$ steadily rises to 1 as γ converges to 0. At this point it jumps to 0 and increases to 0.5 again as γ converges to $-\infty$. In this case, group-size again does not matter because only the minimum contribution counts (Lee (2012)). As evident from the left panel of Figure 4, for the smaller group the efforts are larger over the whole range of γ. Therefore, the changes in the winning probability at $\gamma = 0$ are due to a changing productivity of the larger and the smaller group with γ. This is evident from the right panel of Figure 4, where the impact of group 2 is consistently higher than the one of group 1 for $\gamma < 0$ and vice versa for $\gamma > 0$. The driving force behind these results is thus the CES function which for $\sum a_i^k \neq 1$ changes not only the degree of

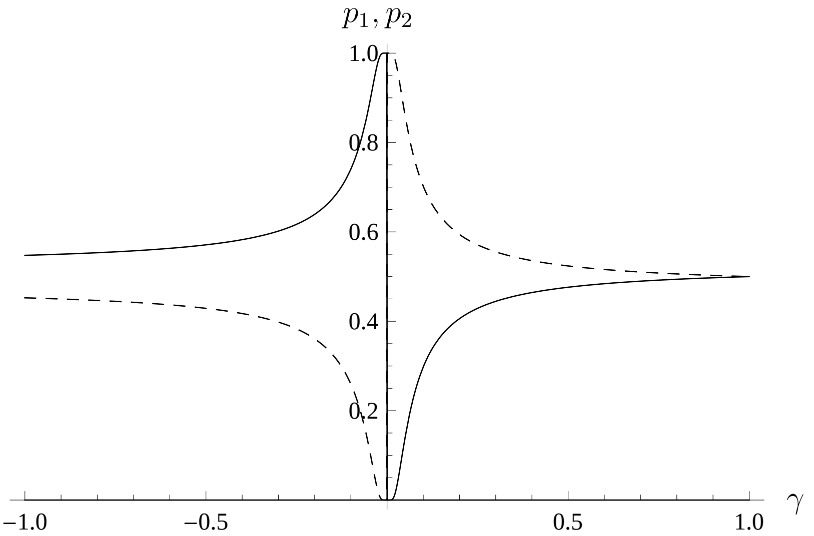

Figure 3: Equilibrium probabilities for different values of γ ($m_1 = 11$, $m_2 = 10$).

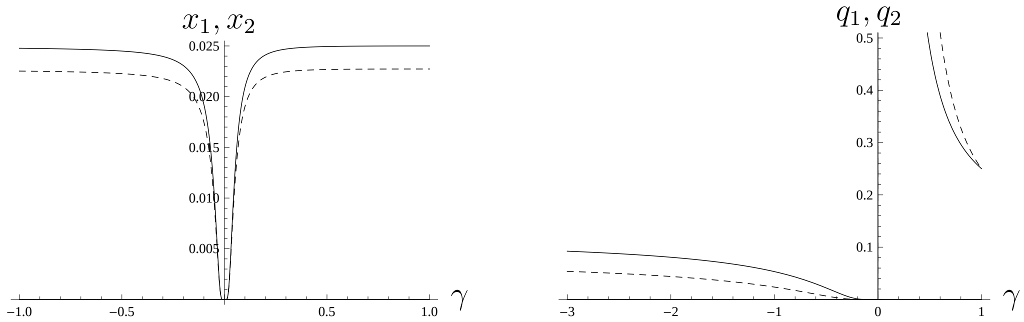

Figure 4: Effort levels (left) and impacts (right) for different values of γ.

complementarity with γ but also the efficiency as becomes apparent when inserting $x_i^k = x_i$ and $\gamma_i = \gamma$ into the impact function:

$$q_i(x_i, ..., x_i) = g_i \cdot x_i \cdot \left(\sum_{k=1}^{m_i} a_i^k \right)^{1/\gamma} . \tag{C.7}$$

Whenever $\sum a_i^k > 1$, the function becomes infinitely large for $\gamma \to 0^+$ and infinitely small for $\gamma \to 0^-$. The rate of convergence depends on the sum of all a_i^k, which was smaller for group 2 in the above case. Therefore it had a disadvantage for positive γ and an advantage for negative γ.

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